RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE AND THE GROWTH FUNCTION OF SUBMANIFOLDS

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1. Let $\overline{M}$ be a Riemannian C^∞-manifold, and let M be a compact C^∞-submanifold of codimension 1, possibly with boundary. If there exists a globally defined C^∞ unit-normal field N on M, we say that M is *relatively orientable*. If $\overline{M}$ is orientable, then M is relatively orientable if and only if M is orientable. We call a submanifold of codiemnsion 1 a *hypersurface*.

Suppose M is relatively orientable, and let N be a C^∞ unit-normal field on M. For $m \in M$, let $g_m(s)$ denote the geodesic (of $\overline{M}$), parametrized by arc length s, such that $g_m(0) = m$ and $\dot{g}_m(0) = N(m)$, where $\dot{g}_m$ is the tangent vector to g_m. Let M_s be the set of points $\{g_m(s) \mid m \in M\}$. For small s, the set M_s is a C^∞-submanifold of $\overline{M}$. Denote the volume of M_s by A(s). Following H. Wu and R. A. Holzsager [3], [4], we call A(s) the *growth function* of M. Let $A^{(k)}$ denote the kth derivative of A with respect to s. Wu and Holzsager [3], [4] showed that the two-dimensional Riemannian manifolds of constant curvature equal to c are characterized by the equation $A^{(2)} + cA = 0$ for all M. Let $L = d/ds$, and let c be a constant. Let

$$L_n = (L^2 + c)(L^2 + 9c)(L^2 + 25c) \cdots (L^2 + (n-1)^2 c) \tag{1}$$

if n is even, and

$$L_n = L(L^2 + 4c)(L^2 + 16c) \cdots (L^2 + (n-1)^2 c) \tag{2}$$

if n is odd. We shall prove the following four theorems.

THEOREM 1. *Suppose $\overline{M}$ is an n-dimensional Riemannian manifold of constant curvature equal to c. Then the growth function A of each compact, relatively orientable hypersurface M of $\overline{M}$ satisfies the differential equation*

$$L_n A = 0.$$

Furthermore, this is the only differential equation of lowest order that A satisfies for every M.

THEOREM 2. *Suppose the growth function A of each compact, relatively orientable hypersurface M of $\overline{M}$ satisfies the differential equation*

$$A^{(3)} + c_2 A^{(2)} + c_1 A^{(1)} + c_0 A = 0, \tag{3}$$

where c_2, c_1, and c_0 are functions of s, and no lower-order differential equation is satisfied by A for all M; then $\overline{M}$ is a three-dimensional Riemannian manifold of constant curvature, say K, and therefore, by Theorem 1, $c_2 = c_0 = 0$ and $c_1 = 4K$.

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