# SINGULAR INNER FUNCTIONS WITH DERIVATIVE IN $B^p$

H. A. Allen and C. L. Belna

For $0 < p < 1$, the space $B^p$ is the class of all functions $f(z)$ analytic in $D = \{z: |z| < 1\}$ for which

$$\|f\|_p = \int_0^1 \int_0^{2\pi} |f(re^{i\theta})| (1 - r)^{1/p-2} d\theta \, dr < \infty .$$

For basic properties of $B^p$, see [4].

A singular inner function is a function of the form

$$S_\mu(z) = \exp \int \frac{z + e^{it}}{z - e^{it}} d\mu(e^{it}) ,$$

where $\mu$ is a positive measure on the unit circle, singular with respect to Lebesgue measure. For a discussion of inner functions, see [3] or [5].

In [1], J. G. Caughran and A. L. Shields asked whether there exists a singular inner function with derivative in the Hardy class $H^{1/2}$. M. R. Cullen [2] conjectured that no singular inner function has derivative in the larger space $B^{1/2}$. In this paper, we disprove the conjecture of Cullen but leave open the question of Caughran and Shields.

THEOREM. *Let the measure* $\mu$ *consist of discrete masses* $a_j$ *such that the sequence* $\{a_j\}_{j=1}^\infty$ *belongs to some space* $\ell^{1/q}$ $(1 < q < \infty)$, *and let*

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad \gamma < \frac{2p}{4p - 1}.$$

*Then* $S'_\mu \in B^\gamma$. *In particular,* $S'_\mu \in B^{1/2}$.

*Proof.* The formula $S_\mu(z) = \exp \sum_{j=1}^\infty a_j \dfrac{z + e^{it_j}}{z - e^{it_j}}$ implies that

$$S'_\mu(z) = S_\mu(z) \sum_{j=1}^\infty \frac{-2a_j e^{it_j}}{(z - e^{it_j})^2}.$$

Since

$$\Re \frac{z + e^{it_j}}{z - e^{it_j}} = 1 - 2 \frac{1 - r\cos(\theta - t_j)}{|z - e^{it_j}|^2} \qquad (z = re^{i\theta}),$$

we have the formula

Received July 22, 1971.

Michigan Math. J. 19 (1972).