NORMAL ANALYTIC FUNCTIONS AND LINDELÖF'S THEOREM

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1. INTRODUCTION

This paper deals with various weakenings of the hypotheses in Lindelöf's classical limit theorem for bounded analytic functions.

Let D and Γ denote the unit disk $|z| < 1$ and the unit circle $|z| = 1$, respectively. The open subarc of Γ with endpoints $z = 1$ and $\zeta = e^{i\theta}$ $(0 < \theta < 2\pi)$ is denoted by $A(0, \theta)$, while the domain bounded by the arc $A(0, \theta)$ and the closed chord subtending $A(0, \theta)$ is denoted by $G(0, \theta)$. For points $\zeta = e^{i\theta}$, we write $\zeta \to 1^+$ if $\theta \to 0^+$.

Lindelöf's theorem [1, p. 42] is the following proposition.

THEOREM L. *Suppose that* f *is a bounded analytic function in* D, *that*

$$\lim_{\zeta \to 1^+} |f(\zeta)| = \lim_{\zeta \to 1^+} (\limsup_{z \to \zeta} |f(z)|) = 0, \tag{1}$$

and that $0 < \theta < 2\pi$. *Then* $f(z) \to 0$ *as* $z \to 1$ *in* $G(0, \theta)$.

It is known [2, Theorem 5.6] that the condition (1) can be replaced by the condition

$$\lim_{\zeta \to 1^+, \zeta \in \Gamma - E} |f(\zeta)| = 0, \tag{2}$$

where $\mu E = 0$ (μ denotes Lebesgue measure on Γ). Moreover, it follows from a theorem of C. Carathéodory (see [1, p. 207] or [2, Theorem 5.5]) that if the radial limits of f satisfy the condition

$$\left|\lim_{r \to 1} f(r\zeta)\right| < \varepsilon$$

for almost every point ζ in some arc $A(0, \theta)$, then

$$|f(\zeta)| \leq \varepsilon \qquad (\zeta \in A(0, \theta)).$$

(The latter inequality can also be deduced from the representation of f by its Poisson integral.) Thus we can replace the condition (1) in Theorem L by the condition

$$\lim_{\zeta \to 1^+, \zeta \in \Gamma - E} (\lim_{r \to 1} f(r\zeta)) = 0, \tag{3}$$

where $\mu E = 0$ and f has a radial limit at each $\zeta \in \Gamma - E$.

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