DENSITY THEOREMS FOR ALGEBRAS OF OPERATORS AND ANNIHILATOR BANACH ALGEBRAS

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INTRODUCTION

Let X be a Banach space, $\mathscr{B}(X)$ the algebra of bounded operators on X, and $\mathscr{F}(X)$ the algebra of bounded operators on X with finite-dimensional range. An algebra A of operators on X is *irreducible* if the only closed A-invariant subspaces of X are $\{0\}$ and X (many authors use "transitive" in place of "irreducible"). In Section 1, we consider conditions on a subalgebra A of $\mathscr{B}(X)$ under which $\mathscr{F}(X) \subset A$. In particular, we show that if X is reflexive and A is a closed irreducible subalgebra of $\mathscr{B}(X)$ that contains a nonzero operator in $\mathscr{F}(X)$, then $\mathscr{F}(X) \subset A$ (Theorem 2).

In Section 2, we prove some Stone-Weierstrass theorems in the setting where B is an annihilator algebra and A is a closed subalgebra satisfying certain conditions (Theorems 9, 10, and 11). The following is a special case of Theorem 10: If B is an annihilator B^*-algebra and A is a semisimple closed subalgebra of B with the property that for each pair of distinct maximal left ideals M and N in B, A contains an element in M but not in N, then A = B. In [7], I. Kaplansky proved a similar theorem with the stronger hypothesis that A is a closed *-subalgebra of B (see [7, Theorem 2.2, p. 223]).

1. CONDITIONS IMPLYING THAT A SUBALGEBRA OF $\mathscr{B}(X)$ CONTAINS $\mathscr{F}(X)$

An algebra A is called *semiprime* if A has no nonzero nilpotent left or right ideals. In fact, A is semiprime if it has no nonzero nilpotent left ideals. For in this case, suppose that R is a nonzero right ideal of A such that $R^n = \{0\}$ and $R^{n-1} \neq \{0\}$, where n is an integer $(n > 1)$. Set $N = R^{n-1}$. Then $N \neq \{0\}$ and $N^2 = \{0\}$. Choose $a \in N$ $(a \neq 0)$. Then $(aA)^2 \subset N^2 = \{0\}$. Therefore $(Aa)^3 = A(aA)^2 a = \{0\}$. By hypothesis, a belongs to the set $\{b \in A \mid Ab = \{0\}\}$; but this set is a nilpotent left ideal of A. This contradiction proves that A has no nonzero nilpotent right ideals.

Throughout this paper, X denotes a Banach space of dimension greater than 1.

LEMMA 1. *Let* A *be an irreducible subalgebra of* $\mathscr{B}(X)$. *Then* A *is semiprime.*

Proof. By the remarks preceding the lemma, it suffices to prove that A has no nonzero nilpotent left ideal. Suppose A contains such a left ideal. Then there exists $T \in A$ $(T \neq 0)$ such that $(AT)^2 = \{0\}$. The set

$$Y = \{x \in X \mid Sx = 0 \text{ for all } S \in A\}$$

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