COHOMOLOGY OF COMPACT MINIMAL SUBMANIFOLDS

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1. INTRODUCTION

Let N be a Riemannian manifold, and let $f: M \to N$ be an immersion. M is said to be *minimal* in N if the mean curvature of M in N is identically zero. In [4], J. Simons studied minimal immersions by considering elliptic differential equations involving cross-sections of various Riemannian vector bundles. In view of his results and the classical relation between harmonic forms and the cohomology of a Riemannian manifold, it is perhaps natural to ask whether there is any connection between minimality and cohomology. In this note we prove the following proposition.

THEOREM. *If* N *is a compact, connected, orientable Riemannian manifold with positive-semidefinite Ricci curvature and* $f: M \to N$ *is a minimal immersion of a compact, connected, orientable manifold* M *such that the image of* M *is not contained in a totally geodesic submanifold of* N, *then the natural map*

$$f^*: H^1(N, \mathbb{R}) \to H^1(M, \mathbb{R})$$

is one-to-one and into.

This result should be compared with the work of T. T. Frankel [1] on minimal hypersurfaces in manifolds with positive-definite Ricci curvature.

2. NOTATION

Let N be a Riemannian manifold with connection $\overline{\nabla}$, and let $f: M \to N$ be an immersion; we shall not in general differentiate between a point p in M and its image in N. There is an orthogonal decomposition $N_p = M_p \oplus M_p^\perp$ with respect to the metric on N. If U is a vector field on N, we shall denote its component tangent to M by U^T, and its component normal to M by U^N. If ∇ is the connection on M with respect to the induced metric, then for tangential vector fields X and Y,

$$\nabla_X Y = (\overline{\nabla}_X Y)^T. \tag{1}$$

If ξ is a normal vector field on M and X is a tangential vector field, define

$$A_\xi X = -(\overline{\nabla}_X \xi)^T. \tag{2}$$

It is well known (see [3, p. 14]) that $(A_\xi X)_p$ depends only on X_p and ξ_p, so that A_{ξ_p} is well-defined and is a symmetric linear operator on M_p. We recall that M is minimal in N if and only if trace $A_{\xi_p} = 0$ for all normal vector fields ξ and all $p \in M$.

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