APPROXIMATIONS OF DOUBLY SUBSTOCHASTIC OPERATORS

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1. INTRODUCTION AND PRELIMINARIES

We prove three approximation theorems for positive contractions T on $L_1[0, \infty)$ having at least one of the two properties $T1 = 1$ and $T^*1 = 1$. The main results concern approximations of such operators by convex combinations of those operators induced by invertible measure-preserving maps on $[0, \infty)$ in well-known operator topologies.

Let X, $\mathfrak{F}$, and μ denote the nonnegative real half-line, the class of Lebesgue measurable sets, and Lebesgue measure. On X, we shall consider only $\mathfrak{F}$-measurable real functions (modulo μ-equivalence), and by a set on X we shall always mean an element of $\mathfrak{F}$. We shall omit the phrase "almost everywhere," it being understood wherever applicable. We assume $1 \leq p \leq \infty$. Let $L_p = L_p(X, \mathfrak{F}, \mu)$, and let $[L_p]$ be the Banach space of bounded linear operators from L_p into itself. We say that T is a *positive contraction* on L_p if $T \in [L_p]$, $Tf \geq 0$ for each f $(0 \leq f \in L_p)$, and $\|T\|_p \leq 1$.

For each positive contraction T on L_1, the adjoint T^* determined by the equation $\int_X (Tf)\, g\, d\mu = \int_X fT^* g\, d\mu$ for $f \in L_1$ and $g \in L_\infty$ is a positive contraction on L_∞. The operators T and T^* can be extended uniquely to positive linear operators on the cone of nonnegative numerical functions u and v as follows:

$$Tu = \lim_n Tf_n, \qquad \text{where } 0 \leq f_n \in L_1,\ f_n \uparrow u,$$

$$T^*v = \lim_n T^*g_n, \qquad \text{where } 0 \leq g_n \in L_\infty,\ g_n \uparrow v.$$

The extensions satisfy the equation $\int_X (Tu)\, v\, d\mu = \int_X uT^* v\, d\mu$. In particular, suppose $T1 \leq 1$. This condition is equivalent to the condition that

$$\int_X T^* g\, d\mu \leq \int_X g\, d\mu$$

for $0 \leq g \in L_1 \cap L_\infty$, and thus T^* is uniquely extended to a positive contraction on L_1. The extended operator will also be denoted by T^*. Observe that in this case T is also a positive contraction on L_∞. We shall always assume that T and T^* represent the extended operators.

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