HELSON SETS IN COMPACT AND LOCALLY COMPACT GROUPS

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We continue our investigation (begun in [1] and [4]) of the measure space $M_0(G)$, where G denotes an infinite, nondiscrete, locally compact group, not necessarily abelian. In the present paper, we show that each measure in $M_0(G)$ is continuous. We further show that if G is compact or metrizable, then a Helson set cannot support a nonzero measure in $M_0(G)$ (a *Helson set* is a compact set P in G such that every continuous function on P can be extended to a function in the Fourier algebra $A(G)$ of the group G).

Let G denote an infinite, nondiscrete, locally compact group (not necessarily abelian) with left-invariant Haar measure m_G, and let $M(G)$ denote the space of finite regular Borel measures on G. We use the notation and machinery developed by P. Eymard [5] as well as that in [2]. Let Σ denote the equivalence classes of the continuous unitary representations on G, and for $\pi \in \Sigma$, let $\mathscr{H}_\pi$ denote the representation space. For $\mu \in M(G)$, we define the function $\hat{\mu}$ on Σ by

$$\pi \mapsto \hat{\mu}_\pi = \int_G \pi(x)\, d\mu(x)\,.$$

For $\mathscr{S} \subset \Sigma$, let

$$\|\mu\|_{\mathscr{S}} = \sup\{\|\hat{\mu}_\pi\|_\infty : \pi \in \mathscr{S}\}\,,$$

where $\|\hat{\mu}_\pi\|_\infty$ denotes the operator norm on $\mathscr{H}_\pi$. We define $C^*(G)$ to be the completion of $L^1(G)$ in $\|\cdot\|_\Sigma$ (see [5, p. 187]). Let $\{\rho\}$ denote the subset of Σ containing just the left-regular representation of G on $L^2(G)$. Let $C^*(G)$ denote the completion of $L^1(G)$ in $\|\cdot\|_\rho$ (see [5, p. 187]). If G is abelian or compact, then $C^*(G) = C^*_\rho(G)$.

If $\mu \in M(G)$, we let $\rho(\mu)$ denote the bounded operator defined on $L^2(G)$ by $h \mapsto \mu * h$ $(h \in L^2(G))$ with operator norm $\|\rho(\mu)\|_\rho$. Let $\mathscr{B}(L^2(G))$ denote the bounded operators on $L^2(G)$. Then $C^*_\rho(G)$ can be identified with the closure in $\mathscr{B}(L^2(G))$ of the set $\rho(L^1(G)) = \{\rho(f): f \in L^1(G)\}$. If G is abelian, then $C^*_\rho(G)$ is isomorphic to the space $C_0(\hat{G})$ of continuous functions on the dual group $\hat{G}$ that vanish at infinity; and if G is compact, then $C^*_\rho(G) \cong \mathscr{C}_0(\hat{G})$ (see [1]).

Let $VN(G)$ denote the von Neumann subalgebra of $\mathscr{B}(L^2(G))$ generated by the left translation operators (see [5, p. 210]). If $\mu \in M(G)$, then $\rho(\mu) \in VN(G)$. Furthermore, we have the inclusion $C^*_\rho(G) \subset VN(G)$. If G is abelian, then $VN(G) \cong L^\infty(\hat{G})$; and if G is compact, then $VN(G) \cong \mathscr{L}^\infty(\hat{G})$ (see [1]).

Let $B(G)$ denote the linear subspace of $C^B(G)$ (the continuous bounded functions on G) spanned by the continuous, positive-definite functions. Then $B(G)$ can be

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