# SPLITTING LOCALLY COMPACT ABELIAN GROUPS

Ronald O. Fulp

The results of this paper constitute a part of a continuing investigation of the splitting problem in the category of locally compact abelian groups. More precisely, assume that $A \overset{\phi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is an extension of A by C in the category $\mathscr{L}$ of locally compact abelian groups (the morphisms of $\mathscr{L}$ are continuous homomorphisms). Under what conditions on A and C does the extension split? The papers [2] and [3] gave partial answers to the question, under various connectivity assumptions on A and C, respectively. In [1], the same problem was studied under the additional assumption that the extension be a pure exact sequence. The present paper is parallel to the development in [3]: we assume that G is a torsion-free group or a torsion group, rather than connected or totally disconnected (these analogues are suggested by Pontryagin's duality theory). Of course, this change of hypothesis greatly changes the problem; but our technique is similar to the technique used for the analogous problems in [1], [2], and [3], in that our investigation is basically homological.

More specifically, we determine the groups G in $\mathscr{L}$ for which the extension

$$G \rightarrowtail Y \twoheadrightarrow X \tag{1}$$

splits for each X in the class $\mathscr{C}$, where $\mathscr{C}$ may denote either the class of all locally compact, abelian torsion groups or the class of all locally compact, torsion-free abelian groups. We also determine the groups H in $\mathscr{L}$ for which the extension

$$X \rightarrowtail Y \twoheadrightarrow H \tag{2}$$

splits for each X in $\mathscr{C}$ with the same two choices of $\mathscr{C}$ as in the last statement.

In [2], we developed considerable homological apparatus for dealing with such problems. Suffice it to say that one may develop the extension functor in the category $\mathscr{L}$ along the lines used by S. MacLane [8] for R-modules. There are topological difficulties, but these are solved in [2]. One uses the usual definition of equivalence of extensions and the usual definition of Baer sum of extensions to make the class of all extensions $A \rightarrowtail B \twoheadrightarrow C$ into a (discrete) group Ext(C, A). The group Ext(C, A) has the usual functorial properties. One point should be mentioned: the definition of an extension is tailored to meet the requirement that if $A \overset{\phi}{\rightarrowtail} B \overset{\theta}{\twoheadrightarrow} C$ is an extension, then $\phi(A)$ is isomorphic to A *in the category* $\mathscr{L}$, and $\theta$ may be identified with the natural mapping $B \twoheadrightarrow B/\phi(A) = C$. In order to make such identifications, we *define* an extension $A \overset{\phi}{\rightarrowtail} B \twoheadrightarrow C$ in $\mathscr{L}$ to be an exact sequence, where $\phi$ is an injective morphism of $\mathscr{L}$, where $\theta$ is a surjective morphism of $\mathscr{L}$, and where both $\phi$ and $\theta$ are required to be open onto their respective images.

Once the group Ext has been constructed, we can rephrase the problems (1) and (2) above as follows: (1) determine the groups G of $\mathscr{L}$ for which Ext(X, G) = 0 for

Received October 30, 1970.
The author gratefully acknowledges partial support from the Engineering Foundation of North Carolina State University during the writing of this paper.

Michigan Math. J. 19 (1972).