PIECEWISE LINEAR EMBEDDINGS OF BOUNDED MANIFOLDS

C. H. Edwards, Jr.

1. INTRODUCTION

Let M^m and Q^q ($q \geq m + 3$) be *bounded* piecewise linear (PL) manifolds (that is, let M and Q be compact and have nonempty boundaries ∂M and ∂Q), and write $d = 2m - q$. Let $f: M \to Q$ be a proper PL mapping (that is, suppose $f^{-1}(\partial Q) = \partial M$). J. F. P. Hudson has proved (Theorem 8.2 of [3]) that f is homotopic, as a map of pairs $(M, \partial M) \to (Q, \partial Q)$, to a proper PL embedding, provided that $(M, \partial M)$ is d-connected and $(Q, \partial Q)$ is $(d + 1)$-connected. This result is similar to the earlier PL embedding theorem of Irwin [5], who assumes instead that $f \mid \partial M: \partial M \to \partial Q$ is an embedding, M is d-connected, and Q is $(d + 1)$-connected, and proves that f is homotopic to a proper embedding *via* a homotopy that is fixed on ∂M. Thus Hudson's theorem deals with embedding *modulo* the boundary, and Irwin's with embedding *relative* to the boundary. (Section 5 contains a remark on the relation between the two types of embedding problem.)

The purpose of this paper is to prove a generalization of Hudson's theorem. We replace the hypothesis that $(Q, \partial Q)$ is $(d + 1)$-connected with the weaker assumption that $(Q, \partial Q)$ is d-connected and $f_*: \pi_{d+1}(M, \partial M) \to \pi_{d+1}(Q, \partial Q)$ is an epimorphism. This is Theorem 1 in Section 3; however the details of the proof require the hypothesis that $q \geq m + 4$ (rather than $q \geq m + 3$).

2. ENGULFING LEMMAS

The proof of Theorem 1 requires several engulfing lemmas. The first two are elementary and well-known, and we omit their proofs.

First a remark on terminology: Suppose that the polyhedron X collapses to the subpolyhedron Y in the bounded PL manifold M. Zeeman [6, Chapter 7] calls $X \searrow Y$ an *interior collapse* if $X - Y$ is contained in int M. *All* collapses in this paper will be interior collapses.

LEMMA 1 (see [3, Lemma 7.1] and [6, Lemma 37]). *Let* X_0, X, Y *be polyhedra in the bounded* PL *manifold* M *such that* $X_0 \subset X$ *and* $X \cap (Y \cup \partial M) \subset X_0$, *and such that* X *collapses to* X_0. *If* U *is a neighborhood of* X_0 *in* M, *then there exists a* PL *homeomorphism* $h: M \to M$ *such that* $X \subset h(U)$ *and* $h \mid X_0 \cup Y \cup \partial Q = \text{identity}$.

LEMMA 2 [3, Lemma 7.3]. *If* X *and* Y *are subpolyhedra of the polyhedron* Z *and* Z *collapses to* X, *then there exists a polyhedron* $T \subset Z$ *such that*

$$X \cup Y \subset X \cup T, \qquad Z \searrow X \cup T \searrow X,$$

and $\dim T \leq \dim Y + 1$.

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