THE CLUSTER SET OF THE PRODUCT OF TWO FUNCTIONS IN H^∞

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Let I denote the set of all inner functions in H^∞, where H^∞ is the Banach algebra of all bounded analytic functions on the open unit disk $D = \{z: |z| < 1\}$. Let I^* denote the set of all functions $f(z)$ in H^∞ for which the cluster set $C(f, e^{i\theta})$ at each point $e^{i\theta}$ on the circumference $C = \{z: |z| = 1\}$ is either the closed unit disk $|w| < 1$ or else a single point of modulus 1. This class of functions has been investigated in several recent papers (see, for example, [2] and [5]). In particular, A. J. Lohwater and G. Piranian [5, Theorem 3] have shown that the class I^* contains an outer function.

In [3], the question was raised whether I^* is a semigroup under multiplication. After some preliminary considerations, we show that I^* is not closed under multiplication (Theorem 2). The technique we use to construct functions in $I^* - I$ leads to several surprising results. For example, we show that the norm of the product of two functions in I^* can be arbitrarily small. In the remainder of the paper, we discuss some of the consequences of Theorem 2 that underscore the differences between inner functions and functions in $I^* - I$.

We begin with a simple fact, which, for purposes of reference, we state without proof as a lemma.

LEMMA 1. *Let* $\{\lambda_n\}$ *be a sequence of nonzero numbers in* D. *If the series* $\sum_{n=1}^\infty |1 - \lambda_n|$ *converges, then*

$$\left|1 - \prod_{n=1}^{\infty} \lambda_n\right| \leq \sum_{n=1}^{\infty} |1 - \lambda_n|.$$

Our next lemma gives an inequality for a Blaschke product whose zeros converge rapidly to C. Let $\{a_n\}$ be a sequence of points in D such that $|a_n| = r_{2n-1}$ $(n = 1, 2, \cdots)$, where

$$r_n = 1 - 2^{-n^2}.$$

Since $\{a_n\}$ is a Blaschke sequence, we can form the associated Blaschke product

$$B(z) = \prod_{k=1}^{\infty} b(a_k, z),$$

where

$$b(a_k, z) = \frac{\bar{a}_k}{|a_k|} \frac{a_k - z}{1 - \bar{a}_k z}.$$

Received April 10, 1971.

Michigan Math. J. 19 (1972).