FINITE-DIFFERENCE INEQUALITIES AND AN EXTENSION OF LYAPUNOV'S METHOD

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1. INTRODUCTION

Many authors have investigated the boundedness and stability properties of differential equations by considering one-sided estimates of solutions. It is natural to expect that an estimate of the lower bound for the rate at which the solutions approach the origin or the invariant set would yield interesting refinements of stability notions. V. I. Zubov's notion of a uniform attractor (see [6] and [7]) is a refinement of such a nature.

The purpose of this paper is to prove some finite-difference inequalities that have a wide range of applications in the study of finite-difference equations. Incorporating an idea used by Zubov in stability theorems [6], [7], we introduce the concepts of mutual stability and boundedness; our two-sided estimates ensure that the motion remains in tube-like domains.

2. DEFINITIONS AND BASIC THEOREMS

Let G be an open set in an n-dimensional vector space R^n with norm $\|x\|$, let I denote the set of nonnegative integers, and let the function $h(k, x)$: $I \times G \to R^n$ be continuous in x, for each k. Consider the finite-difference equation

$$\Delta x(k) = h(k, x(k)), \tag{2.0}$$

where x and h can be scalars or vectors, $h(k, 0) = 0$, and $\Delta x(k) = x(k+1) - x(k)$.

A function $x(k) = x(k; k_0, x_0)$ is called a solution of the difference equation (2.0) if it satisfies the three conditions:

(a) $x(k; k_0, x_0)$ is defined for $k_0 \leq k \leq k_0 + \beta$, for some positive integer β or for all $k \geq k_0$,

(b) $x(k_0; k_0, x_0) = x_0$ (we call this the *initial vector*),

(c) $\Delta x(k; k_0, x_0) = h(k, x(k; k_0, x_0))$ for $k_0 \leq k \leq k_0 + \beta - 1$ or for all $k \geq k_0$.

Hereafter, we assume that a solution to (2.0) exists and is uniquely defined for all $k \geq k_0$ by the initial vector x_0, and that this solution is continuous with respect to the initial vector x_0. More specifically, we assume that if $\{x_n\}$ is a sequence of vectors with $x_n \to x_0$ as $n \to \infty$, then the solutions through x_n converge to the solution through x_0:

$$x(k; k_0, x_n) \to x(k; k_0, x_0) \quad \text{as } n \to \infty.$$

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