IDEALS OF COMPACT OPERATORS ON HILBERT SPACE

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This paper is dedicated in sorrow to the memory of our late colleague, David Topping.

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite-dimensional Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the ring of bounded linear operators on $\mathscr{H}$. The two-sided ideals in $\mathscr{L}(\mathscr{H})$ were originally described by J. von Neumann (see, for instance, [2, Section 1]), and since then a good deal of attention has been devoted to a special class of such ideals, namely, the *norm ideals* of R. Schatten [4] (also called *s. n. ideals* by I. C. Gohberg and M. G. Kreĭn [3]). Very little seems to be known, however, about the general ideal structure of $\mathscr{L}(\mathscr{H})$. For example, it is well known (and easy to prove) that every proper ideal in $\mathscr{L}(\mathscr{H})$ is contained in the ideal $\mathfrak{C}$ of compact operators; consequently, if T is an operator on $\mathscr{H}$ and if there exists a proper ideal $\mathfrak{J}$ in $\mathscr{L}(\mathscr{H})$ such that T belongs to $\mathfrak{J}$, then T is certainly compact. But, given a compact operator, can we assert that there exist ideals other than $\mathfrak{C}$ that contain it? The purpose of this note is to make a modest beginning toward a general theory of ideals. In particular, we answer the preceding question in the affirmative.

In the sequel, all Hilbert spaces are understood to be complex and separable, and all operators are bounded and linear. Moreover, the term *ideal* will always be used to mean two-sided ideal. We remind the reader that if $\mathfrak{J}$ is an ideal, and if T belongs to $\mathfrak{J}$, then T^* and $|T| = (T^*T)^{1/2}$ also belong to $\mathfrak{J}$; moreover, every proper ideal $\mathfrak{J}$ is not only contained in the ideal $\mathfrak{C}$ of all compact operators, but also contains the ideal $\mathfrak{F}$ of all operators of finite rank. The von Neumann-Calkin characterization of the ideals in $\mathscr{L}(\mathscr{H})$ goes as follows. Let C denote the collection of all the nonnegative real sequences $\{\lambda_n\}_{n=1}^{\infty}$ that tend to zero as n tends to infinity. Following Calkin, we call a subset J of C an *ideal set* if it satisfies the following conditions.

(i) If $\{\lambda_n\} \in J$ and if π denotes any permutation of the positive integers, then $\{\lambda_{\pi(n)}\} \in J$.

(ii) If $\{\lambda_n\}$ and $\{\mu_n\}$ both belong to J, then so does $\{\lambda_n + \mu_n\}$.

(iii) If $\{\lambda_n\} \in J$, and if $\{\mu_n\}$ is any sequence in C such that $\mu_n \leq \lambda_n$ for every n, then $\{\mu_n\} \in J$.

Now let $\mathfrak{J}$ be any proper ideal in $\mathscr{L}(\mathscr{H})$. If T belongs to $\mathfrak{J}$, then so does $|T|$, and since $|T|$ is compact, it has an orthonormal basis of eigenvectors. If the corresponding eigenvalues are arranged into a sequence (counting multiplicities), that sequence belongs to C. In this way a sequence in C, determined up to permutation, is associated with each operator T in $\mathfrak{J}$. It turns out that the set of all sequences so

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