RIESZ POTENTIALS, k, p-CAPACITY, AND p-MODULES

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1. INTRODUCTION

Let R^m denote m-dimensional Euclidean space with points $x = (x_1, x_2, \cdots, x_m)$ and Euclidean norm $|x|$. For $p \geq 1$, we denote by $\|f\|_p$ the L^p-norm of f taken over the whole space R^m. Let $s = (s_1, s_2, \cdots, s_m)$ be a multi-index with length $|s| = \sum s_i$, and let $D^s f$ be the corresponding derivative of f of order $|s|$. As usual, C_0^∞ is the class of all infinitely differentiable functions with compact support. Finally, k is a positive integer, and F is a compact subset of R^m.

A measure of the size of a set F is given by the k, p-capacity of F, which we define as follows.

Definition 1. The k, p-*capacity of* F is

$$\Gamma_{k,p}(F) = \inf_f \sum_{|s| \leq k} \|D^s f\|_p^p,$$

where the infimum is taken over all $f \in C_0^\infty$ with $f \geq 1$ on F.

We get the same class of null-sets if in the definition we require all the functions f to have support in some fixed neighbourhood O of F. In fact, if $\phi \in C_0^\infty$ has support in O and $\phi = 1$ on F, then $f\phi$ has support in O, $f\phi \geq 1$ on F, and

$$\sum_{|s| \leq k} \|D^s(f\phi)\|_p \leq \text{const.} \sum_{|s| \leq k} \|D^s f\|_p,$$

where the constant does not depend on f.

We also get the same class of null-sets if in the sum in the definition we take $|s| = k$ instead of $|s| \leq k$ (if $kp \geq m$, we must then assume that the support of f is a subset of a fixed sphere). This may be proved by means of inequalities of Sobolev type.

For k = 1, the notion of k, p-capacity was used by Serrin [4] in the investigation of removable singularities of partial differential equations. It has also been used in the theory of quasiconformal mappings in R^m (Gehring [3]).

By the *Riesz potential of order* α $(0 < \alpha < m)$ *of the function* f (or the α-potential of f) we shall mean the function U_α^f defined by

$$U_\alpha^f(x) = \int \frac{f(y)\,dy}{|x - y|^{m-\alpha}}.$$

The purpose of this paper is to prove the following theorem.

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