INTERTWINING ANALYTIC TOEPLITZ OPERATORS

James A. Deddens

Let A and B be bounded linear operators on Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, respectively. We say that a bounded linear operator X from $\mathscr{H}$ into $\mathscr{K}$ *intertwines* A and B if XA = BX. B. Sz.-Nagy and C. Foiaş [8] have shown that any intertwining operator between two contractions extends to an intertwining operator between their coisometric extensions (see also [3]). However, an intertwining operator between two subnormal operators need not extend to an intertwining operator between their normal extensions (example: $A = T_z$ and $B = 0$; see also [1]). R. G. Douglas and C. Pearcy [2] gave a necessary and sufficient condition that there be no nonzero intertwining operator between two normal operators; because of the theorem of Fugledge and Putnam, this condition is symmetric in the two operators. However, there exist operators for which this property is not symmetric. That is, there exist operators A and B such that there are nonzero operators intertwining A and B but no nonzero operators intertwining B and A (example: $A = T_z$ and $B = 0$). These two phenomena make the study of intertwining operators between analytic Toeplitz operators of interest. In this note, we obtain an asymmetric, sufficient condition for the nonexistence of nonzero intertwining operators between two analytic Toeplitz operators. By means of this result, we then obtain an example of an operator whose commutant is abelian but that does not have a cyclic vector.

For convenience, we consider H^2 to be the Hilbert space of analytic functions in the unit disk for which the functions $f_r(\theta) = f(re^{i\theta})$ are bounded in the L^2-norm, and H^∞ to be the linear manifold of bounded functions in H^2. For $\phi \in H^\infty$, T_ϕ (or $T_{\phi(z)}$) is the *analytic Toeplitz operator* on H^2 defined by the relation $(T_\phi f)(z) = \phi(z)f(z)$. We shall denote the spectrum of T_ϕ by $\sigma(T_\phi)$ and the set $\{\phi(z): |z| < 1\}$ by range (ϕ). Then $\sigma(T_\phi)$ = closure (range (ϕ)) [4, Problems 26 and 197]. If $\phi \in H^2$, then the function $\bar{\phi}$ defined by $\bar{\phi}(z) = \overline{\phi(\bar{z})}$ is also in H^2. For $|\lambda| < 1$, define $h_\lambda \in H^2$ by the relation $h_\lambda(z) = (1 - \lambda z)^{-1}$. Then $T_\phi^* h_\lambda = \bar{\phi}(\lambda) h_\lambda$ for $\phi \in H^\infty$ [7].

LEMMA. *If* Λ *is an uncountable subset of the disk* $|\lambda| < 1$, *then* $\{h_\lambda: \lambda \in \Lambda\}$ *spans* H^2.

Proof. Suppose $f \in H^2$ is orthogonal to $\{h_\lambda: \lambda \in \Lambda\}$. Then $f(\bar{\lambda}) = (f, h_\lambda) = 0$ for $\lambda \in \Lambda$. If Λ is uncountable, then $f \equiv 0$, since f is analytic. Hence $\{h_\lambda: \lambda \in \Lambda\}$ spans H^2.

THEOREM. *Let* $\phi, \psi \in H^\infty$. *If* range $(\psi) \not\subseteq \sigma(T_\phi)$, *then the only bounded linear operator* X *satisfying the condition* $XT_\phi = T_\psi X$ *is* $X = 0$.

Proof. Let $N \equiv$ range $(\psi) \cap \mathbb{C} \setminus \sigma(T_\phi)$. Then N is either a nonempty open set or a singleton, depending on whether range (ψ) is an open set or a singleton (that is, whether ψ is nonconstant or constant). In either case, $\psi^{-1}(N)$ is a nonempty *open* subset in $\{z: |z| < 1\}$, and hence uncountable. By our lemma, $\{h_\lambda: \bar{\lambda} \in \psi^{-1}(N)\}$

Received July 7, 1970.
This research was partially supported by the National Science Foundation.

Michigan Math. J. 18 (1971).