A MINKOWSKI AREA HAVING NO CONVEX EXTENSION

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Let $A = E^4 \wedge E^4$, and let M be the set of all simple (decomposable) elements in A. Let $\{\lambda_i\}$ denote a set of nonnegative numbers whose sum is 1. Let h be a real-valued function on M. Then [3, p. 20] h is *weakly convex* on M if

$$h(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 h(x_1) + \lambda_2 h(x_2)$$

whenever $x_1, x_2 \in M$ and $\lambda_1 x_1 + \lambda_2 x_2 \in M$. Also, h is *convex* on M if

(a) $$h\left(\sum_{i=1}^{k} \lambda_i x_i\right) \leq \sum_{i=1}^{k} \lambda_i h(x_i)$$

whenever $\{x_1, \cdots, x_k\}$ is a finite set of points in M and $\sum_{i=1}^{k} \lambda_i x_i \in M$, and

(b) there exists a linear function L on A with $L \leq h$ on M.

Suppose that h is convex on M, and let $q(x) = \inf \sum_{i=1}^{k} \lambda_i h(x_i)$, where the infimum is taken over all k-tuples $\{x_i\}_1^k$ of points in M and over all $\{\lambda_i\}_1^k$ such that $x = \sum_{i=1}^{k} \lambda_i x_i$. Then (see [3, p. 21]) q extends h to A and is convex.

Let K be a central convex body in E^4 with its center at the origin. If $R \in M$ and $\mathscr{R}$ is the plane determined by R, let $f(R) = |R|/e(K \cap \mathscr{R})$, where $e(K \cap \mathscr{R})$ is the Euclidean area of $K \cap \mathscr{R}$. It is known that the Minkowski area f is weakly convex [4, p. 62]. We shall show that, for suitable K, f is not convex; this answers a question of Busemann and Petty [2, Problem 10]. This problem was discussed in greater detail in [4] and listed again in [3, p. 33].

Let $r^i = (r_1^i, r_2^i)$ $(i \in I = \{1, 2, 3, 4\})$ be linear functions on E^2 that are linearly independent. Let p_1 and p_3 in E^2 be determined by the equations $r^1(p_1) = r^2(p_1) = 1$ and $r^2(p_3) = r^3(p_3) = 1$. Then it is not difficult to verify that, except possibly for sign, the area of the parallelogram spanned by p_1 and p_3 is

$$\frac{[1\,2] + [2\,3] + [3\,1]}{[1\,2]\,[2\,3]}, \qquad \text{where } [i\,j] = \det \begin{vmatrix} r_1^i & r_2^i \\ r_1^j & r_2^j \end{vmatrix}.$$

Thus, if P is a symmetric octagon whose consecutive sides, in appropriate order, are

$$-r^4 = 1, \quad r^1 = 1, \quad r^2 = 1, \quad r^3 = 1, \quad r^4 = 1, \quad -r^1 = 1, \quad -r^2 = 1, \quad -r^3 = 1,$$

then area $P = A = A_1 + A_2 + A_3 + A_4$, where

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