THE COVERING THEOREM FOR UPPER BASIC SUBGROUPS

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All groups considered in this paper are primary abelian groups. We follow the notation and terminology of [1]. Recall that a subgroup B of G is a *basic subgroup* of G if it has the following properties:

(i) B is a direct sum of cyclic groups,

(ii) B is pure in G,

(iii) G/B is divisible.

Let r(G) denote the rank of a group G, and set

$$r_G = \min\{r(G/B)\colon B \text{ is a basic subgroup of } G\}.$$

If B is a basic subgroup of G such that $r(G/B) = r_G$, then B is called an *upper* basic subgroup of G. As L. Fuchs has mentioned in [1, p. 105], upper basic subgroups are important because an upper basic subgroup B of G, together with r(G/B), may reveal much more information about the structure of G than can be conveyed by an arbitrary basic subgroup.

A. R. Mitchell [6] has proved the following for reduced p-groups. If B_1 and B_2 are upper basic subgroups of G_1 and G_2, respectively, then $B_1 + B_2$ is an upper basic subgroup of $G_1 + G_2$. If B is an upper basic subgroup of a high subgroup H of G, then B is also an upper basic subgroup of G. Left open in [6] was the question whether each basic subgroup is contained in an upper basic subgroup. In his review of Mitchell's paper, in the *Zentralblatt* (166, p. 292), P. Grosse stated "whereas the author could not prove that a basic subgroup B of G is contained in an upper basic subgroup whenever G is a reduced p-group he paved the way to this (hopefully correct) statement." In this paper we settle the question affirmatively, but travel a different (unpaved) route. The solution comes as an immediate corollary to a structure theorem (Theorem 3) that gives almost complete information about basic subgroups that are not upper in relation to upper basic subgroups. The main result of the paper, however, is the following. *If* G *is a direct sum of cyclic groups and if* B *and* B' *are basic subgroups of* G *such that* $G/B \cong G/B'$, *then there exists an automorphism* π *of* G *such that* $\pi(B) = B'$. We cast this result in slightly more general form (Theorem 1). The proof involves extending height-preserving automorphisms on subgroups; for related results, see [2] and [3].

THEOREM 1. *Let* G *be a primary group, and let* $G = G_0 + H$, *where* H *is a direct sum of cyclic groups. Suppose that* B_0 *is a basic subgroup of* G_0. *Let* $B = B_0 + A$ *and* $B' = B_0 + A'$ *be basic subgroups of* G. *There exists an automorphism* π *of* G *that is the identity on* G_0 *and maps* B *onto* B' *if and only if*

$$\text{(I)} \qquad G/\{G_0, B\} = G/\{G_0, B'\}.$$

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