INEQUALITIES GOVERNING THE OPERATOR RADII ASSOCIATED WITH UNITARY ρ-DILATIONS

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1. INTRODUCTION

Our purpose in this paper is to present some results concerning the operator radii $w_\rho(T)$. First we recall the pertinent definitions. Suppose T is an operator on a Hilbert space $\mathscr{H}$ (in what follows, all Hilbert spaces are complex, and all operators are bounded and linear). We say the operator T belongs to the class C_ρ $(0 < \rho < \infty)$ if there exists a unitary operator U on some Hilbert space $\mathscr{K}$ such that $\mathscr{K}$ contains $\mathscr{H}$ as a subspace and such that $T^n h = \rho P_{\mathscr{H}} U^n h$ for all $h \in \mathscr{H}$ $(n = 1, 2, \cdots)$. B. Sz.-Nagy and C. Foiaş introduced the classes C_ρ in [9] to provide a unified framework for two results that we may state as follows: (i) (see Sz.-Nagy [7]) $T \in C_1$ if and only if $\|T\| \leq 1$; (ii) (see C. A. Berger [1]) $T \in C_2$ if and only if $w(T) \leq 1$, where $w(T)$ denotes the numerical radius of T, that is,

$$w(T) = \sup \{ |(Th, h)| : h \in \mathscr{H} \text{ and } \|h\| = 1 \} .$$

In our paper [5], we defined the operator radii $w_\rho(\cdot)$ $(0 < \rho < \infty)$ by the equation

$$w_\rho(T) = \inf \{ u : u > 0 \text{ and } u^{-1} T \in C_\rho \} .$$

Independently, J. P. Williams used the same functions in [11]. The family of operator radii $w_\rho(\cdot)$ includes the familiar operator norms $\|\cdot\|$ $(= w_1(\cdot))$ and $w(\cdot)$ $(= w_2(\cdot))$. We may adjoin the other well-known operator radius, namely, the spectral radius $\nu(\cdot)$, to this family in a natural way: the relation

$$\lim_{\rho \to \infty} w_\rho(T) = \nu(T)$$

holds, so that we are led to define $w_\infty(T)$ as $\nu(T)$.

These and other known properties of the classes C_ρ and the functions $w_\rho(\cdot)$ are described carefully in Section 2. Sections 3, 4, and 5 contain results that we believe to be new. Experience suggests that $w_\rho(T)$ may be a convex function of ρ (for fixed T) in every case. In Section 3, we obtain results in this direction. For example, we show that if $0 < \rho_1, \rho_2 < 2$ and $F(\rho) = (w(T))^{-1}$, then

$$F((\rho_1 + \rho_2)/2) \geq \lambda F(\rho_1) + (1 - \lambda) F(\rho_2) ,$$

where $\lambda = (2 - \rho_1)(2 - \rho_1 + 2 - \rho_2)^{-1}$. From this inequality, we deduce that the function $(2 - \rho)(w_\rho(T))^{-1}$ is increasing on $(0, 1]$. Combining this result with the same inequality, we show that $w_\rho(T)$ is indeed convex in the range $(0, 1]$. Section 4 contains convexity results of a less precise nature; there we simply demonstrate the existence of certain convexity constants. In Section 5, we apply the earlier

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