A CHARACTERIZATION OF ZERO SETS FOR A^∞

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1. INTRODUCTION

Let U and T denote the open unit disk $\{z \mid |z| < 1\}$ and the unit circle $\{z \mid |z| = 1\}$. The space A^∞ consists of all nonconstant analytic functions f in U such that for each positive integer n, the n^{th} derivative $f^{(n)}$ is bounded in U.

The following characterization of the possible zero sets for functions in this class is due to B. A. Taylor and D. L. Williams [3].

THEOREM 1.1 (Taylor and Williams). *In order that a closed subset* Z *of* $\overline{U} = U \cup T$ *be the zero set of a function in* A^∞, *it is necessary and sufficient that* (a) *the set* $Z \cap U = \{r_k e^{i\theta_k}\}_{k=1}^\infty$ *satisfy the condition*

$$\sum_{k=1}^{\infty} (1 - r_k) < \infty, \tag{1.1}$$

and (b)

$$\int_{-\pi}^{\pi} \log \operatorname{dist}(e^{i\theta}, Z)\, d\theta > -\infty. \tag{1.2}$$

The main result of this paper provides an intrinsic characterization of such zero sets; at the same time, it yields a shorter proof of Theorem 1.1.

THEOREM 1.2. *Let* Z *be a closed subset of* $\overline{U}$, *and put* $E_1 = Z \cap T$. *Then* Z *is the zero set of a function in* A^∞ *if and only if* (a) *holds and*

(c) $E_2 = E_1 \cup \{e^{i\theta_k}\}_{k=1}^\infty$ *is a Carleson set.*

This result clearly depends only on the radial projection of Z on T, modulo the Blaschke condition (1.1). Our next theorem gives a condition under which (c) is always satisfied.

THEOREM 1.3. *Suppose* Z *is a closed subset of* $\overline{U}$ *satisfying* (a) *and that* $E = Z \cap T$ *is a Carleson set. If*

$$\sum_{k=1}^{\infty} [\operatorname{dist}(e^{i\theta_k}, E)]^\alpha < \infty \tag{1.3}$$

for some $\alpha \geq 1$, *then* Z *is the zero set of a function in* A^∞.

The hypothesis of Theorem 1.3 appears in [4], where it is shown only that Z is the zero set of a function whose derivative belongs to H^1.

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