A RECURSIVE FUNCTION, DEFINED ON A COMPACT INTERVAL AND HAVING A CONTINUOUS DERIVATIVE THAT IS NOT RECURSIVE

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We shall construct such a function f by placing -shaped bumps at each number of the form 2^{-n}, where n belongs to a recursively enumerable, nonrecursive set $\mathscr{A}$, and by leaving the neighborhood of all other numbers 2^{-n} flat. For $n \in \mathscr{A}$, the slope of the graph at 2^{-n} can be effectively bounded from below, given n. Thus, if we could compute $f'(2^{-n})$ recursively, we could decide whether $n \in \mathscr{A}$, contradicting the nonrecursiveness of $\mathscr{A}$.

We first define the function f nonconstructively, and then prove that it is actually recursive.

Let

$$\theta(x) \equiv \begin{cases} x(x^2 - 1)^2 & \text{for } -1 \leq x \leq 1, \\ 0 & \text{for } |x| > 1. \end{cases}$$

Then $\theta(x)$ has the required form on $[-1, 1]$, and

$$\theta(-1) = \theta(0) = \theta(1) = 0, \qquad \theta'(-1) = \theta'(1) = 0, \qquad \theta'(0) = 1.$$

The function θ takes its minimum value $-\lambda$ at $x = -1/\sqrt{5}$ and its maximum $+\lambda$ at $x = +1/\sqrt{5}$. We call θ on $[-1, +1]$ a *bump* of length 2 and height λ. Now we define bumps $\theta_{\alpha\beta}$ of length 2α and height β.

The function $\theta_{\alpha\beta}(x) \equiv (\beta/\lambda)\, \theta(x/\alpha)$ satisfies the conditions

$$\theta_{\alpha\beta}(-\alpha) = \theta_{\alpha\beta}(0) = \theta_{\alpha\beta}(\alpha) = 0, \qquad \theta'_{\alpha\beta}(-\alpha) = \theta'_{\alpha\beta}(\alpha) = 0, \qquad \theta'_{\alpha\beta}(0) = \theta/\lambda\alpha,$$

$$-\beta \leq \theta_{\alpha\beta}(x) \leq \beta \qquad (-\alpha \leq x \leq \alpha).$$

For each $n \in \mathscr{A}$, we shall put a bump $\theta_{\alpha_n \beta_n}$ around 2^{-n}; that is, we define f(x) as follows:

If $n \in \mathscr{A}$ and $\delta \in [-\alpha_n, +\alpha_n]$, then $f(2^{-n} + \delta) \equiv \theta_{\alpha_n \beta_n}(\delta)$. Otherwise, $f(x) \equiv 0$. The parameters α_n, β_n for $n \in \mathscr{A}$ will be defined by

$$\alpha_n \equiv 2^{-k-2n-2}, \qquad \beta_n \equiv 2^{-k-n-2},$$

where $n = h(k)$ and h is a function enumerating $\mathscr{A}$ without repetitions.

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