LOCALLY COMPACT LATTICES WITH SMALL LATTICES

Tae Ho Choe

In [4], L. Anderson asked whether each locally compact, connected topological lattice has a base consisting of open sublattices. We shall show that this question has a negative answer even in a compact, connected, metrizable distributive lattice. However, we shall see that if a lattice has finite dimension (either codimension of H. Cohen [9] or inductive dimension of Urysohn and Menger), then it has such a base. The following natural question arises: What is a necessary and sufficient condition for a lattice to have such a base? In the first section, we shall answer this question. We shall then prove that no locally compact, connected, complemented lattice has a base consisting of open sublattices. This implies that each locally compact, relatively complemented lattice that is either finite-dimensional or has a base of open sublattices is totally disconnected. J. Lawson [11] studied the parallel problem for a semilattice. He proved that locally compact, locally connected, finite-dimensional semilattices have small semilattices.

The following theorem was conjectured by A. D. Wallace [14], and it was proved in [2] and [7]: If L is a compact, connected lattice of codimension at most n, then the number of elements in its center, denoted by Card(Cen(L)), is at most 2^n. In the second section, we shall see that this theorem also holds in a locally compact, connected lattice with 0 and 1. Furthermore, if the lattice is not compact, then $\text{Card}(\text{Cen}(L)) \leq 2^{n-1}$.

For a pair of subsets A and B of a topological lattice L, we use $A \wedge B$ and $A \vee B$ to denote the sets

$$\{a \wedge b \mid a \in A \text{ and } b \in B\} \quad \text{and} \quad \{a \vee b \mid a \in A \text{ and } b \in B\},$$

respectively. For a subset A of L, we let A^*, A°, and $F(A) = A^* \setminus A^\circ$ denote the closure, the interior, and the boundary of A, respectively. All other terms and definitions used in this paper are the same as in [3] and [7]. It is known ([1], [3], and [5]) that every locally compact, connected lattice is chain-wise connected, locally convex, and locally connected.

1. LATTICES WITH SMALL LATTICES

A topological lattice that has a base consisting of open sublattices is called a lattice with small lattices. Recently, J. Lawson [12] gave an example of a compact, connected, metrizable, distributive lattice L that admits no nontrivial lattice-homomorphism into the unit interval I with the usual order, that is, every lattice-homomorphism of L into I is a constant mapping. We show that this lattice has no base consisting of open sublattices. Suppose that the lattice L has such a base. Then, by [13, Theorem 5], the topology of L must be the interval topology of L. By [13, Theorem 6], L admits enough lattice-homomorphisms to separate points of L.

Received December 16, 1969.

This research was supported by NRC grant A4809 and by the Department of University Affairs, Ontario, Canada.

Michigan Math. J. 18 (1971).