ON WEYL'S THEOREM

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Let $\mathscr{B}(H)$ be the algebra of all bounded operators on an infinite-dimensional complex Hilbert space H, and let $\mathscr{K}$ be the closed ideal of compact operators. L. Coburn [3] has defined the Weyl spectrum $\omega(A)$ by

$$\omega(A) = \bigcap \sigma(A + K),$$

where $\sigma(A)$ denotes the spectrum of A in $\mathscr{B}(H)$ and the intersection is taken over all K in $\mathscr{K}$. A celebrated theorem of H. Weyl [9] asserts that if A is normal, then $\omega(A)$ consists precisely of all points in $\sigma(A)$ except the isolated eigenvalues of finite multiplicity.

In [3], Coburn proved that Weyl's theorem holds for two large classes of generally nonnormal operators, namely, the class of hyponormal operators and the class of Toeplitz operators. In this paper, we shall show that Weyl's theorem holds for yet another class of operators.

Recall that an operator A is a Fredholm operator if it has a closed range and both a finite-dimensional kernel and cokernel. The class $\mathscr{F}$ of Fredholm operators constitutes a multiplicative open semigroup in $\mathscr{B}(H)$. In fact [1], if π is the natural quotient map from $\mathscr{B}(H)$ to $\mathscr{B}(H)/\mathscr{K}$, then A is in $\mathscr{F}$ if and only if $\pi(A)$ is invertible. For any A in $\mathscr{F}$, the index i(A) is defined by the formula

$$i(A) = \dim[\ker A] - \dim[\operatorname{coker} A],$$

and it is known that i is a continuous integer-valued function on $\mathscr{F}$.

Let L^2 and L^∞ denote the Lebesgue spaces of square-integrable and essentially bounded functions with respect to normalized Lebesgue measure on the unit circle in the complex plane. Let H^2 and H^∞ denote the corresponding Hardy spaces. If $\phi \in L^\infty$, the Toeplitz operator induced by ϕ is the operator T_ϕ on H^2 defined by $T_\phi f = P(\phi f)$; here P stands for the orthogonal projection in L^2 with range H^2. Recall that the linear span $H^\infty + C$ of H^∞ and C is a closed subalgebra of L^∞ [5, Theorem 2], where C stands for the space of continuous, complex-valued functions on the unit circle. This algebra can also be characterized as the subalgebra of L^∞ generated by H^∞ and the function $\bar{z}$. It is well-known [4] that if $\phi \in H^\infty + C$, then T_ϕ is a Fredholm operator if and only if ϕ is an invertible function of $H^\infty + C$.

The relation between the index and the invertibility of Toeplitz operators is described by the following result of Coburn [3].

LEMMA A. *If* $\phi \in L^\infty$, *then either* $\ker T_\phi = (0)$ *or* $\operatorname{coker} T_\phi = (0)$.

For ϕ in L^∞, let $R(\phi)$ denote the essential range of ϕ. Suppose that $u \in H^\infty + C$, that $|u| = 1$ a.e., and that T_u is invertible; it is easy to show that then the spectrum $\sigma(T_u)$ of T_u is $R(u)$. In fact, we can use the same argument as in

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