A NOTE ON MULTIVALUED MONOTONE OPERATORS

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1. INTRODUCTION

Let E and F be two real vector spaces in duality with respect to a bilinear form $\langle x, u \rangle$ for $x \in E$ and $u \in F$. A (generally multivalued) mapping $T: E \to F$ is called a *monotone operator* if

$$\langle x - y, u - v \rangle \geq 0$$

whenever $u \in Tx$ and $v \in Ty$; the *domain* of T is defined by

$$D(T) = \{x \in E; Tx \text{ nonempty}\}.$$

The purpose of this note is to show, roughly, that a monotone operator that is actually multivalued admits no continuous selection (Proposition 1) and is not lower-semicontinuous (Proposition 3). We give applications to duality mappings (Proposition 2) and to subdifferentials of convex functions (Proposition 4).

2. SELECTION

A *selection* for a multivalued mapping $T: E \to F$ is a (singlevalued) mapping $\tilde{T}: D(T) \to F$ such that $\tilde{T}x \in Tx$ for every $x \in D(T)$. A selection $\tilde{T}$ is said to be *hemicontinuous* at $x \in D(T)$ if it is continuous (in the $\sigma(F, E)$-topology of F) at x, on each line segment in $D(T)$ with endpoint x.

We shall say that a point x of a subset D of E is *quasi-internal* to D if the convex cone generated by the set of y for which the line segment $[x, y]$ is contained in D is $\sigma(E, F)$-dense in E. Thus each internal point of D, or each point of D if D is a $\sigma(E, F)$-dense subspace of E or an open subset of E (for some vector-space topology on E), is quasi-internal to D.

PROPOSITION 1. *Let* $T: E \to F$ *be a monotone operator that is not single-valued at* $x \in D(T)$. *If* x *is quasi-internal to* $D(T)$, *then* T *admits no selection that is hemicontinuous at* x.

Proof. Suppose that T admits a selection $\tilde{T}: D(T) \to F$, hemicontinuous at x. Since T is not singlevalued at x, there exists $u \in Tx$ with $u \neq \tilde{T}x$. Take y such that $x + ty \in D(T)$ for all $t \in [0, 1]$. The monotonicity of T implies that

$$\langle (x + ty) - x, \tilde{T}(x + ty) - u \rangle \geq 0 \qquad \forall\, t \in [0, 1],$$

so that

$$\langle y, \tilde{T}(x + ty) - u \rangle \geq 0 \qquad \forall\, t \in\,]0, 1];$$

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