ON THE POISSON-STIELTJES REPRESENTATION FOR FUNCTIONS WITH BOUNDED REAL PART

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1. INTRODUCTION

The main purpose of this paper is to establish results that connect a Poisson-Stieltjes integral with boundary properties of the function it represents. A well-known theorem of Herglotz [5, p. 196] states that a function f holomorphic in $|z| < 1$ with positive real part has a Poisson-Stieltjes representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t) + i\, \Im f(0), \tag{1.1}$$

where μ is a nondecreasing function of bounded variation on $[-\pi, \pi]$. Briefly, we shall say that f is a *Herglotz* function with *mass distribution* μ if (1.1) holds. For such functions, Fatou's theorem [6, p. 46] shows that $\Re[f]$ has angular limit at e^{it_0} equal to $\mu'(t_0)$ wherever the derivative exists (including $\mu'(t_0) = +\infty$). We shall seek other relationships that connect f and its mass distribution μ.

THEOREM 1. *If μ is the mass distribution for a Herglotz function* f, *and* $\sup \Re[f] < \infty$, *then μ is nondecreasing and absolutely continuous and has bounded Dini derivates.*

In fact, every difference quotient of μ is bounded by the bounds on $\Re[f]$. Example 1 shows that f as distinguished from $\Re[f]$ may nevertheless be unbounded.

Further information about μ is obtained under the condition

$$\iint_G |f'(\sigma)|^2\, d\sigma < \infty, \tag{1.2}$$

where G is a domain of the form $\{|z| < 1\} \cap \{|z - \zeta| < r\}$, $|\zeta| = 1$. The integral represents the area of f(G) on the Riemann surface associated with f. Condition (1.2) has been used extensively in the boundary theory of conformal mapping [1]. We shall say that f has the *finite-area property* at ζ ($|\zeta| = 1$) if (1.2) holds for some $r > 0$ (for brevity, we occasionally write $f \in \mathrm{FAP}(\zeta)$). The usefulness of this condition arises from the fact that if

$$G_n = \{|z| < 1\} \cap \{r_{n+1} < |z - \zeta| < r_n\}$$

and $r_n \to 0$ as $n \to \infty$, then

$$\iint_{G_n} |f'(\sigma)|^2\, d\sigma = o(1) \quad (n \to \infty). \tag{1.3}$$

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