A CHARACTERIZATION OF TWO-DIMENSIONAL RIEMANNIAN MANIFOLDS OF CONSTANT CURVATURE

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Let $\overline{M}$ be a Riemannian manifold, and let M be a *compact hypersurface,* that is, a compact orientable submanifold of codimension 1 of $\overline{M}$, possibly with boundary. (Everything is assumed to be C^∞.) For sufficiently small s, let M_s denote the set of points lying on geodesics normal to M (and on a fixed side of M) at distance s from M. Denoting the volume of M_s by $\mathscr{A}(s)$, we call the real-valued function $\mathscr{A}$ (defined in a neighborhood of zero) the *growth function* of M. In [1], it is shown that $\mathscr{A}$ is a polynomial of degree at most 1, for each compact hypersurface in $\overline{M}$, if and only if $\overline{M}$ is locally isometric to $\mathbb{R}^2$. The purpose of the present note is to point out that the technique employed in [1] actually allows us to prove the following theorem, which is more general and more satisfactory.

THEOREM. *A Riemannian manifold has the property that the growth function* $\mathscr{A}$ *of each one of its compact hypersurfaces satisfies the linear differential equation*

$$\mathscr{A}'' + c\mathscr{A} = 0 \tag{1}$$

(*where* c *is a fixed constant*) *if and only if it is a two-dimensional Riemannian manifold of constant curvature equal to* c.

Using the known facts about the solutions of equation (1), we may rephrase the theorem in an equivalent way: the two-dimensional Riemannian manifolds of constant zero curvature are characterized by the fact that their growth functions are polynomials of degree at most 1; the two-dimensional Riemannian manifolds of constant positive curvature c are characterized by the fact that their growth functions are expressible as linear combinations of $\cos\sqrt{c}\,s$ and $\sin\sqrt{c}\,s$; and the two-dimensional Riemannian manifolds of constant negative curvature are characterized by the fact that their growth functions are expressible as linear combinations of $\cosh\sqrt{-c}\,s$ and $\sinh\sqrt{-c}\,s$.

Before giving the proof of the theorem, we must recall the results proved in [1]. We let M be a compact hypersurface of $\overline{M}$, and we let M_s be as above. Denoting by Ω_s the volume form of M, we have by definition the relation

$$\mathscr{A}(s) = \int_{M_s} \Omega_s. \tag{2}$$

To state the formula for $\mathscr{A}''(s)$, we separate our discussion into two cases.

Case 1: $\dim \overline{M} = 2$. In this case, each M_s is simply a finite C^∞-curve. Let K denote the curvature function of the surface $\overline{M}$. Then

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