HEREDITARY LOCAL RINGS

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Professor P. M. Cohn recently provided an example of a right principal ideal domain with prime factorization that is not a left hereditary ring [1], and we were curious about the pathologies of a ring that allow this to happen.

In this note, we investigate some of the conditions under which a ring is a principal ideal domain. The treatment of the local case is easiest. Our results (see the lemma, and Corollaries 1 and 2) are closely related to those of G. A. Probert [5, Theorems 6.4 and 6.5]. We also obtain a result in the semilocal case (Proposition 1) that is similar to results obtained by G. O. Michler [3, Lemma 3.4 and Corollary 3.10]. (Our results are not obtainable from the results of Probert and Michler, nor do they imply them.) Our final remarks deal with the relations between principal ideal domains and rings all of whose proper residue rings are quasi-Frobenius.

By a *local ring* (R, M) we shall mean a ring R with a unique maximal ideal M that is both a left maximal ideal and a right maximal ideal.

By a *semilocal ring* we shall mean a ring R with finitely many maximal ideals $M_1, \cdots, M_t$, each being both a left maximal ideal and a right maximal ideal.

Recall that in an Ore domain, $Ra \cap Rb \neq 0$ whenever a and b are nonzero elements of R, and that a left principal ideal domain is both a left hereditary ring and a left Ore domain.

We shall also use freely the result that a projective module over a local ring is a free module [2].

1. THE SEMILOCAL CASE

PROPOSITION 1. *Let* R *be a semilocal ring. If* $M_i = Rm_i = m_i R$ *and* $\bigcap_{i=1}^{\infty} M_j^i = 0$ *for* $j = 1, \cdots, t$, *then* R *is a left and right principal ideal ring.*

Observe that the result is unambiguous, since the assumptions are symmetric.

If J is not nilpotent - in which case R is an Artinian ring - then R is a domain.

Proof. By straightforward reasoning, one finds that $M_i \cap M_j = M_i M_j$ for $i \neq j$ and $1 \leq i, j \leq t$. Also,

$$M_{j_1}^{i_1} \cap \cdots \cap M_{j_s}^{i_s} = M_{j_1}^{i_1} \cdot \cdots \cdot M_{j_s}^{i_s}$$

for every set $(i_1, \cdots, i_s)$ of nonzero integers whenever $p \neq q$ implies that $j_p \neq j_q$ $1 \leq j_1, \cdots, j_s \leq t$. Let

$$K = M_{j_1}^{i_1} \cap \cdots \cap M_{j_s}^{i_s}.$$

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