SOME TOPOLOGICAL INVARIANTS OF STONE SPACES

Roger Wiegand

Most of the work of this paper was motivated by a still unsolved problem in homological algebra: Let J be an ideal in the commutative regular ring R, and let U[J] be the corresponding open subset of the maximal ideal space X(R). Is the projective dimension of J a topological invariant of U[J]? It can be shown that J is projective if and only if U[J] satisfies the equivalent conditions of Theorem 2.2. More generally, the projective dimension of J is at most equal to the cohomological dimension of U[J], which in turn is at most equal to the covering dimension of U[J]. In Sections 2 and 5, we show by examples that the covering dimension of U[J] may be strictly greater than the cohomological dimension of U[J]; but no example is known to the author in which the cohomological dimension of U[J] is strictly greater than the projective dimension of J. Most of the paper is concerned with a purely topological investigation of the covering dimension and cohomological dimension of Stone spaces, that is, of open subsets of Boolean spaces. (Stone spaces may be characterized equivalently as locally compact, totally disconnected Hausdorff spaces.)

1. COVERING DIMENSION

Let $\mathcal{U}$ be an open cover of the topological space X. The *order* $o(\mathcal{U})$ is the largest integer n such that there exist $n+1$ distinct members of $\mathcal{U}$ with nonempty intersection. If no such integer exists, we say $o(\mathcal{U}) = \infty$. The *covering dimension* cov dim X is the least integer n such that every open cover of X has an open refinement of order at most n; if no such integer exists, we set cov dim $X = \infty$. Suppose X is a Boolean space, that is, a compact, totally disconnected Hausdorff space. It is well known (and easily proved) that each open cover of X has a finite, disjoint refinement consisting of compact, open sets. We have the following analogue for Stone spaces:

LEMMA 1.1. *Let* X *be a Stone space, and assume* X *has a compact, open cover of order* n. *Then each open cover of* X *has a compact, open refinement of order at most* n. *In particular,* cov dim $X \leq n$.

Proof. Let $\mathcal{U}$ be an open cover of X. Choose a compact, open cover $\mathcal{V}$ of order n, and for each $V \in \mathcal{V}$, let $\mathcal{W}_V$ be a (finite) disjoint, compact, open cover of V that refines $\mathcal{U}$. Then $\mathcal{W} = \bigcup \{\mathcal{W}_V\colon V \in \mathcal{V}\}$ is a compact, open refinement of $\mathcal{U}$, and $o(\mathcal{W}) \leq n$.

THEOREM 1.2. *Let* X *be a Stone space. Then* cov dim $X \leq n$ *if and only if* X *has a compact, open cover of order at most* n.

Proof. One implication follows from Lemma 1.1. To prove the converse, let $\mathcal{U}$ be a compact, open cover of X, and let $\mathcal{V} = \{V_i\colon i \in I\}$ be an open refinement of order at most n. Since each V_i has compact closure and $\mathcal{V}$ is point-finite, an easy argument (similar to [3, Problem 5.V]) based on Zorn's lemma shows that $\mathcal{V}$ can be shrunk to a compact, open cover $\mathcal{W} = \{W_i\colon i \in I\}$, with $W_i \subseteq V_i$ for each $i \in I$. Clearly, $o(\mathcal{W}) \leq n$.

Received January 27, 1969.
This research was supported in part by N.S.F. Grant GP-8751.