ASYMPTOTIC AND INTEGRAL CLOSURE OF ELEMENTS IN MULTIPLICATIVE LATTICES

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1. INTRODUCTION

The cancellation property of ideals (if AB = AC and A $\neq$ 0, then B = C) in a Dedekind domain does not generalize to arbitrary commutative rings. P. Samuel [7] showed how to restore the cancellation property in Noetherian rings by replacing the properties of the ideals themselves with asymptotic properties of their high powers. For this replacement, the notion of asymptotic closure of an ideal is fundamental. In this paper we give a lattice-theoretic characterization of asymptotic closure, we generalize the cancellation law to (commutative) multiplicative lattices that satisfy the ascending chain condition (Theorem 1), and we investigate some properties of this closure operation. In particular, we give a lattice-theoretic characterization of the notion of integral closure of an ideal (in a Noetherian ring), and by means of E. W. Johnson's A-transforms of a Noether lattice [3] we show that the asymptotic closure of an element in a Noether lattice coincides with its integral closure (Theorem 3). Since not all Noether lattices are lattices of ideals of a Noetherian ring [1], Theorem 3 extends M. Nagata's result that the asymptotic and integral closure operations coincide in Noetherian rings [4].

Finally, we establish some relations between the asymptotic closure operation in a Noether lattice and its A-transforms.

2. ASYMPTOTIC CLOSURE IN MULTIPLICATIVE LATTICES

A *multiplicative lattice* is a complete lattice provided with a commutative, associative, join-distributive multiplication for which the greatest element, denoted by I, is also the multiplicative identity (0 denotes the null element). In this section, L denotes a multiplicative lattice that satisfies the ascending chain condition. We shall now generalize the asymptotic closure of ideals (as characterized by D. Rees in [6]) to multiplicative lattices. The development toward the cancellation law is patterned after [5].

Let R denote the ordered additive group of real numbers, together with an element ∞ that satisfies the relations $\alpha + \infty = \infty$, $\infty + \infty = \infty$, $\infty > \alpha$ (here α denotes a real number). A mapping v: L $\rightarrow$ R is a *pseudovaluation* on L if

a) $v(0) = \infty$,

b) $v(I) = 0$,

c) $v(AC) \geq v(A) + v(C)$ $\quad$ (A, C $\in$ L), and

d) $v(A \vee C) \geq \min(v(A), v(C))$ $\quad$ (A, C $\in$ L).

Received September 13, 1968.

This research constitutes a portion of a doctoral dissertation written under the direction of Professor Louis J. Ratliff at the University of California at Riverside (June 1967). The author is indebted to Professor Ratliff for guidance and encouragement.