REPRESENTATIONS OF SOLVABLE LIE ALGEBRAS

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Ado's theorem [1] states that every finite-dimensional Lie algebra over a field of characteristic zero has a faithful finite-dimensional representation. The proofs of this theorem in the literature at the present time do not appear to provide a bound for the dimension of the representation space.

In a forerunner to the theorem of Ado, G. Birkhoff proved that a finite-dimensional, nilpotent Lie algebra has a faithful representation whose space has dimension not greater than $1 + n + n^2 + \cdots + n^{k+1}$, where n is the dimension of the algebra and k is the nilpotency class [2].

The main result of this paper is that every finite-dimensional, solvable Lie algebra over an algebraically closed field of characteristic zero has a faithful representation whose space has dimension not greater than $1 + n + n^n$, where n is the dimension of the algebra. In the nilpotent case, we obtain the bound $1 + n^k$, where k is the nilpotency class. This is sharper than the bound of Birkhoff.

In particular, Section 1 contains a proof of Birkhoff's theorem about nilpotent Lie algebras. We also prove that the enveloping associative algebra determined by the representation has the same nilpotency class as the original algebra. In the second part of this section, we prove that if $\mathscr{L}$ is a finite-dimensional Lie algebra that can be written as a semidirect sum $\mathscr{L} = \mathscr{L}_1 + N$ (N a nilpotent ideal), then Birkhoff's representation of N can be extended to $\mathscr{L}$.

Section 2 deals with splittable Lie algebras. Here we prove that every solvable Lie algebra $\mathscr{L}$ of finite dimension over an algebraically closed field of characteristic zero can be embedded in a solvable algebra of given dimension, determined by the dimension of $\mathscr{L}$ and the dimension of the nilradical of $\mathscr{L}$.

Finally, in Section 3 we obtain a faithful representation of an n-dimensional solvable Lie algebra $\mathscr{L}$ over an algebraically closed field of characteristic zero. This is done by embedding $\mathscr{L}$ in a solvable splittable algebra $\mathscr{L}_1$, as in Section 2, and constructing a representation of $\mathscr{L}_1$ from the results of Section 1.

1. PROPOSITION 1. *Let* $\mathscr{L}$ *be an* n*-dimensional nilpotent Lie algebra with lower central series* $\mathscr{L} = \mathscr{L}^1 \supset \mathscr{L}^2 \supset \cdots \supset \mathscr{L}^k \supset \mathscr{L}^{k+1} = 0$. *Then* $\mathscr{L}$ *is isomorphic to a Lie algebra* $\mathscr{A}$ *of linear transformations of a vector space* M *of dimension not greater than* $1 + n^k$. *Moreover, the product of any* $k + 1$ *elements of* $\mathscr{A}^*$ *is zero.*

Proof. We first choose a basis $x_1, \cdots, x_n$ in the following way:

$$x_1, \cdots, x_{p(1)} \text{ is a basis for } \mathscr{L}^k,$$

$$x_1, \cdots, x_{p(2)} \text{ is a basis for } \mathscr{L}^{k-1},$$

$$\cdots,$$

$$x_1, \cdots, x_{p(k)} = x_n \text{ is a basis for } \mathscr{L}.$$

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