A NOTE ON MANIFOLDS WHOSE HOLONOMY GROUP IS A SUBGROUP OF $Sp(n)\cdot Sp(1)$

Alfred Gray

Recently, several authors [2], [3], [5] have studied quaternionic analogues of Kähler manifolds. There are two possible definitions of such a manifold. Let M be a connected Riemannian manifold of dimension 4n. Then one can require that the holonomy group H(M) be a subgroup of either Sp(n) or $Sp(n)\cdot Sp(1)$, where $Sp(n)\cdot Sp(1) = Sp(n) \times Sp(1)/(\pm \text{identity})$.

The condition $H(M) \subseteq Sp(n)$ is equivalent to the existence on M of parallel, globally defined, almost complex structures I, J, and K that satisfy $IJ = -JI = K$. At first glance, it appears that the condition $H(M) \subseteq Sp(n)$ is the more natural generalization of the notion of Kähler manifold. However, it turns out that if $H(M) \subseteq Sp(n)$, then M has Ricci curvature zero. For this reason there are no known examples of compact Riemannian manifolds that are not flat and satisfy the condition $H(M) \subseteq Sp(n)$.

In this paper we consider Riemannian manifolds with $H(M) \subseteq Sp(n)\cdot Sp(1)$, and we call them *quaternionic Kähler manifolds.* Examples are the quaternionic projective spaces and several other symmetric spaces [5]. We show that this definition of quaternionic Kähler manifolds is equivalent to another, which states that a certain tensor field Q is parallel.

The notion of *quaternionic Kähler submanifold* is defined analogously to that of Kähler submanifold. However, the theory of the former is much simpler than that of the latter, because every quaternionic Kähler submanifold is totally geodesic (Theorem 5). This shows, for example, that the quaternionic analogue of the theory of algebraic varieties is trivial.

First we need the following fact about $Sp(n)\cdot Sp(1)$.

PROPOSITION 1. $Sp(n)\cdot Sp(1)$ *is a maximal Lie subgroup of* SO(4n), *for* $n > 1$.

Proof. Let G_0 be a compact connected Lie subgroup of SO(4n) that contains $Sp(n)\cdot Sp(1)$. Then G_0 is transitive on the unit sphere S^{4n-1}, because $Sp(n)\cdot Sp(1)$ is transitive on S^{4n-1}. If

$$Sp(n)\cdot Sp(1) \subset G_0 \subset SO(4n) \qquad \text{(strict inclusion)},$$

it follows from the classification of connected Lie groups acting transitively and effectively on spheres, that the only possibilities for G_0 are U(n), Spin (7), and Spin (9). We rule out U(n), because Sp(n) is a maximal subgroup of U(n). It is easy to verify that the inclusion $Sp(2)\cdot Sp(1) \subseteq Spin(7)$ is impossible, because both groups have rank 3. Finally, Spin (9) is eliminated because $\dim(Spin(9)) = 36 < 39 = \dim(Sp(4)\cdot Sp(1))$. We conclude that $G_0 = Sp(n)\cdot Sp(1)$.

Received August 16, 1968.
This work was partially supported by National Science Foundation grant GP-8623.