WEAKLY FLAT SPHERES

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1. INTRODUCTION

In [7], D. R. McMillan used the cellularity criterion to give a sufficient condition for the complementary domains of a topologically embedded (n - 1)-sphere in the n-sphere S^n to be open n-cells. In general, if $\Sigma^k \subset S^n$ is a topologically embedded k-sphere, one may ask for conditions that guarantee that the complement $S^n - \Sigma^k$ is homeomorphic to the complement of the standard k-sphere in S^n. In other words, when is $S^n - \Sigma^k$ homeomorphic to $S^{n-k-1} \times R^{k+1}$? When this homeomorphism occurs, we follow Rosen [9] and say that Σ^k is *weakly flat.*

For $k \geq 0$, let D^k be the standard k-cell in Euclidean space R^k. If X is a space, a *loop* in X is a continuous function from ∂D^2 into X. The loop f: $\partial D^2 \to X$ is *null homotopic* if f has a continuous extension F: $D^2 \to X$. In this paper, we study weak flatness *via* the following generalization of the cellularity criterion.

Definition. Let X be a closed set in the interior of a manifold M. We say that M - X *is* 1 - ℓc *at* X if each open neighborhood U of X in M contains an open neighborhood V of X such that each loop in V - X is null homotopic in U - X.

In Section 2, we give an argument similar to that of L. C. Siebenmann in [10] to show that, for $n \geq 5$ and $2 \leq k \leq n - 3$, $\Sigma^k \subset S^n$ is weakly flat if and only if $S^n - \Sigma^k$ is 1 - ℓc at Σ^k. Section 3 is devoted to a proof that under certain conditions, if X is a compact ANR in S^n, if S^n - X is 1 - ℓc at X, and if Y is obtained from X by the deletion of open cones, then S^n - Y is 1 - ℓc at Y. In Section 4 we apply the results in Sections 2 and 3 to questions about weak flatness and cellularity. For example, we show that with dimensional restrictions the boundary of a cellular k-cell in S^n is a weakly flat sphere and that weak flatness is in a certain sense transitive (Theorem 4.1). Finally, in Section 5 we give an example to show that a weakly flat sphere need not be locally flat at any point.

Often we shall indicate the dimension of a space by a superscript the first time it appears in the discussion, and omit the superscript thereafter. We abbreviate *piecewise linear* to PL, throughout. "$X \approx Y$" is to be read as "X *is homeomorphic to* Y" if X and Y are spaces, and as "X *is isomorphic to* Y" if X and Y are groups. "$X \approx_{PL} Y$" means "X *is* PL *homeomorphic to* Y." If X is a subset of a manifold, $N_\varepsilon(X)$ denotes the open ε-neighborhood of X.

2. A CRITERION FOR WEAK FLATNESS

THEOREM 2.1. *Suppose* $\Sigma^k \subset S^n$ *is a topologically embedded* k-*sphere* ($n \geq 5$, $2 \leq k \leq n - 3$). *Then* Σ^k *is weakly flat if and only if* $S^n - \Sigma^k$ *is* 1 - ℓc *at* Σ^k.

Received February 2, 1968.

Most of the results in this paper are contained in the author's doctoral dissertation, which was written under the direction of Professor C. H. Edwards, Jr. at the University of Georgia. The research in this paper was supported by a National Science Foundation Graduate Fellowship.