UNIVALENT FUNCTIONS f(z) FOR WHICH zf'(z) IS SPIRALLIKE

M. S. Robertson

Let $\mathscr{F}$ denote the class of functions F(z) that are regular, univalent, and spirallike in the unit disk $E = \{z: |z| < 1\}$ and that are normalized so that $F(0) = 0$ and $F'(0) = 1$. L. Špaček [5] showed that these functions are characterized by the condition that for some real constant α ($|\alpha| < \pi/2$),

$$\Re\left\{e^{i\alpha}\frac{z\,F'(z)}{F(z)}\right\} > 0 \qquad (z \in E).$$

We denote the corresponding subclasses of $\mathscr{F}$ by $\mathscr{F}_\alpha$; in particular, $\mathscr{F}_0$ is the class of starlike functions. If $F(z) \in \mathscr{F}_0$, then the function $f(z) = \int_0^z \frac{F(t)}{t}dt$ maps E onto a convex domain, and

$$\Re\left(1 + \frac{z f''(z)}{f'(z)}\right) = \Re\,\frac{z[z f'(z)]'}{z f'(z)} = \Re\,\frac{z\,F'(z)}{F(z)} > 0 \qquad (z \in E).$$

In this note, we consider another family of functions that includes the class of convex functions as a proper subfamily. For $-\pi/2 < \alpha < \pi/2$, we say that $f(z) \in S_\alpha$ provided

(i) f(z) is regular in E, $f(0) = 0$, and $f'(0) = 1$,

(ii) $f'(z) \neq 0$ in E,

(iii) $\Re\left(e^{i\alpha}\left(1 + \frac{z f''(z)}{f'(z)}\right)\right) > 0 \qquad (z \in E).$

We note that the three conditions are precisely the conditions for the function $z f'(z)$ to belong to the class $\mathscr{F}_\alpha$. The class S_0 consists of the normalized convex functions.

For general values α ($-\pi/2 < \alpha < \pi/2$), a function in S_α need not be univalent in E. For example, the function

$$f(z) = i(1 - z)^i - i = z + \cdots$$

belongs to the class $S_{\pi/4}$, but it has a zero at each of the points $1 - e^{-2n\pi}$ ($n = 0, 1, \cdots$), and in fact it assumes every value lying on the circle $|w + i| = 1$ infinitely often on the open segment (0, 1) of the real axis. (J. Krzyż and Z. Lewandowski were the first to point out that if $z f'(z)$ is spirallike, the function f(z) is not necessarily univalent; see [2].) However, we shall show that for a certain set of values of α, all functions in S_α are univalent in E.

The situation is analogous to a problem recently considered by P. L. Duren, H. S. Shapiro, and A. L. Shields [1] and by W. C. Royster [4]: for what values of a complex constant α is the function

Received August 2, 1968.

The author acknowledges support from the National Science Foundation (NSF-GP-7439).