EXTREMAL LENGTH AND p-CAPACITY

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1. INTRODUCTION

In Euclidean n-space E_n, consider two disjoint closed sets C_0 and C_1, where C_0 is assumed to contain the closure of the complement of some closed n-ball B. We follow [12] in defining the p-capacity $(1 \leq p < \infty)$ of the pair (C_0, C_1) as

$$(1) \qquad \Gamma_p(C_0, C_1) = \inf \left\{ \int_{E^n} |\text{grad } u|^p \, dL_n \right\},$$

where the infimum is taken over all continuous functions u on E_n that are infinitely differentiable on $E_n - (C_0 \cup C_1)$ and assume boundary values 0 on C_0 and 1 on C_1. Serrin found this notion useful in connection with the question of removable singularities of solutions to certain partial differential equations. The case of conformal capacity is represented when p = n, and it has been fundamental in the development of a theory of quasiconformal mappings in E_n (see [7]). The importance of conformal capacity in the theory of quasiconformal mappings is partly due to an equality of Gehring [6] that relates conformal capacity to the reciprocal of the n-dimensional extremal length of all continua in E_n that intersect both C_0 and C_1. Gehring's proof is valid for a similar equality that involves p-capacity and p-dimensional extremal length, provided $p > n - 1$. It is the purpose of this paper to provide a proof for $p \geq 1$, thus answering in the affirmative question 16 of [13]. We note that the proof is elementary in the sense that it demands only a few basic facts of real function theory. Together with [4, Theorem 7], the result yields a new proof of a theorem of Wallin [14], which relates p-capacity to potential-theoretic capacity. On the other hand, our result, along with that of Wallin, establishes Fuglede's theorem for compact sets, in case k = 1.

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2. NOTATION AND PRELIMINARIES

L_n and H^k will denote n-*dimensional Lebesgue measure* and k-*dimensional Hausdorff measure* in E_n (for properties of the latter, see [2]). If A is an L_n-measurable subset of E_n, let $\mathscr{L}^p(A)$ be the class of functions f for which $|f|^p$ is integrable, and let $\|f\|_p$ be the $\mathscr{L}^p$-norm.

2.1. A real-valued function u defined on an open subset G of E_n is called *absolutely continuous in the sense of Tonelli on* G (ACT) if it is ACT on every interval $I \subset G$ [11, p. 169]. The gradient of u (which will now be denoted by ∇u) exists L_n-almost everywhere on G; moreover, it can easily be shown that the infimum appearing in the definition (1) of p-capacity is not diminished if we extend it to the class of ACT functions that assume the specified boundary values (see [5]).

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