A NONLINEAR PROBLEM IN POTENTIAL THEORY

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1. In this paper we study the following nonlinear boundary-value problem in the unit disc:

$$(1)\qquad \begin{cases} \Delta u + g(x, y, u) = 0 & ((x, y) \in A = [x^2 + y^2 < 1]), \\ u = 0 & ((x, y) \in \partial A = [x^2 + y^2 = 1]), \end{cases}$$

where g is a measurable function of x, y, u satisfying, for some given constants $R_1 > 0$, $R_2 \geq 0$, $L \geq 0$, the inequalities

$$|g(x, y, u)| \leq R_2 \quad \text{for almost all } (x, y) \in A \text{ and for } |u| \leq R_1,$$

$$(2)\qquad |g(x, y, u_1) - g(x, y, u_2)| \leq L\,|u_1 - u_2| \quad \text{for almost all } (x, y) \in A$$

$$\text{and for } |u_1|, |u_2| \leq R_1 .$$

We prove that if g satisfies certain additional inequalities limiting its values and its growth with respect to u, then problem (1) has at least one solution $u(x, y)$ $((x, y) \in A)$ such that

(i) $u(x, y)$ is continuous in $A \cup \partial A$ and is zero on ∂A,

(ii) $u(x, y)$ has first-order partial derivatives that are continuous in A,

(iii) Δu, in the sense of the theory of distributions, is a measurable essentially bounded function,

(iv) Δu satisfies (1) a.e. in A.

If g is also sufficiently smooth in (x, y), then u has continuous second-order partial derivatives and (1) holds everywhere in A in the strict sense. The conditions concerning the growth of g are not unreasonably strict. For instance, for the problem

$$\Delta u + f(x, y)\,|u| = h(x, y) \qquad ((x, y) \in A),$$

$$u = 0 \qquad ((x, y) \in \partial A),$$

all that we require of the measurable functions f and h is that they are bounded and that $|f(x, y)| < 4.13$ in A. The example shows that the present requirement concerning the growth of g is far removed from the usually very strict requirements that are necessary in the use of perturbation techniques.

For the above problem in nonlinear partial differential equations, we apply here a process that we discussed in some generality in [2] and [4] and that has been studied, applied, and extended in a number of ways (see [1], [3], [5], [6], [8], [10],

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