ON THE GROWTH OF UNIVALENT FUNCTIONS

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1. RESULTS

1.1. We shall assume that the function

$$f(z) = z + a_2 z^2 + \cdots \tag{1.1}$$

is analytic and univalent in $|z| < 1$. Throughout the paper, cap denotes logarithmic capacity.

THEOREM 1. *Let* $A(R) = \{z: |f(z)| \geq R\}$. *Then*

$$\operatorname{cap} A(R) \leq 1/\sqrt{R} \qquad (R > 0). \tag{1.2}$$

Furthermore, either

$$\sqrt{R} \operatorname{cap} A(R) \to 0 \qquad (R \to \infty), \tag{1.3}$$

or there is a starlike univalent function $g(z) = z + \cdots$ *such that*

$$\left|\log \frac{f(z)}{g(z)}\right| \leq K\,|\log(1 - |z|)|^{1/2} \qquad (|z| < 1) \tag{1.4}$$

for some constant K.

Since $|\arg g(z)/z| < \pi$, it follows from (1.4) that

$$|\arg f(z)/z| \leq K_1\,|\log(1 - |z|)|^{1/2} \qquad (|z| < 1).$$

A slight modification of the proof of Theorem 1 also gives the bound

$$|f(z)| \leq K_2^{1/\lambda}\,|g(z)|^{1-\lambda}(1 - |z|)^{-2\lambda} \qquad (|z| < 1,\ 0 < \lambda \leq 1).$$

The function $f_m(z) = z(1 - z^m)^{-2/m}$ is univalent in $|z| < 1$, and it has the property

$$\liminf_{R \to \infty} \sqrt{R} \operatorname{cap} A_m(R) > 2^{-2/m} \qquad (m = 1, 2, \cdots).$$

Hence (1.2) is essentially best possible. A result of W. K. Hayman and P. B. Kennedy [5] shows that the right-hand side of (1.4) cannot be replaced by anything smaller than $o(|\log(1 - |z|)|^{1/2})$.

1.2. Hayman was the first to prove a regularity theorem of the following kind: Rapid growth of a univalent function implies rather regular behavior. Let

$$M(r) = \max_{|z|=r} |f(z)|.$$

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