ZEROS OF PARTIAL SUMS OF POWER SERIES

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1. INTRODUCTION

Let $\mathscr{F}$ denote the family of functions that are analytic in the unit disk $|z| < 1$ but not in any disk $|z| < 1 + \varepsilon$ $(\varepsilon > 0)$. If $f(z) = \sum a_k z^k$ belongs to $\mathscr{F}$, we write

$$S_n(z) = S_n(z; f) = \sum_{k=0}^{n} a_k z^k,$$

and we denote by $\rho_n(f)$ the largest of the moduli of the zeros of the polynomial S_n. We write

$$\rho(f) = \liminf_{n \to \infty} \rho_n(f) \qquad \text{and} \qquad P = \sup_{f \in \mathscr{F}} \rho(f).$$

In 1906, M. B. Porter [3] proved that $1 \leq \rho(f) \leq 2$ for all $f \in \mathscr{F}$. Porter showed that his lower bound for $\rho(f)$ is best possible, but he made no similar claim for his upper bound. Quite recently, it has been shown that the constant 2 is *not* best possible. J. Clunie and P. Erdös [1] proved that $P < 2$. In the other direction, they constructed an example to show that $P > \sqrt{2}$. Determination of the exact value of P remains an open problem [2, Problem 7.7].

In the present paper, I prove that $1.7 < P \leq 12^{1/4} = 1.861 \cdots$. The method used to obtain the upper bound is essentially a refinement of the method used by Clunie and Erdös. The lower bound is derived from the remarkably simple example

$$g(z) = \frac{1 + iz - iz^2 - z^3}{1 + z^4}.$$

In Section 3, I prove that $\rho(g) > 1.7$ and indicate why the choice of g is not entirely fortuitous.

2. THE UPPER BOUND

LEMMA. *If* $0 \leq x < 12^{-1/4}$, *then*

$$\sum_{k=1}^{\infty} x^{k+1} |e^{ik\theta} - 1| < 1$$

for all real numbers θ.

Proof. From the Cauchy-Schwarz inequality we get the estimate

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