ON THE MAXIMUM DEGREE IN A RANDOM TREE

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1. INTRODUCTION

A *tree* is a connected graph that has no cycles (see, for example, Ore [7] as a general reference on graph theory). The *degree* of a node x in a tree is the number $d(x)$ of edges joining x to other nodes. Since any tree T_n with n labelled nodes has exactly $n - 1$ edges, it follows that the average value of $d(x)$, as x runs over the nodes of T_n, is $2(1 - 1/n)$. Let $D = D(T_n)$ denote the maximum degree of nodes in the tree T_n, that is, let $D(T_n) = \max\{d(x): x \in T_n\}$. Our object here is to derive, by elementary and crude arguments, an asymptotic formula for the average value of $D(T_n)$ over the set of the n^{n-2} trees T_n with n labelled points.

2. PRELIMINARY RESULTS

We first list some results that we shall use later. (In what follows, n and k will always denote integers such that $1 \leq k \leq n - 1$.)

LEMMA 1. *If the integers* $d(i)$ $(i = 1, 2, \cdots, n)$ *form a decomposition of* $2(n - 1)$, *then there exist*

$$\binom{n-2}{d(1) - 1, \cdots, d(n) - 1}$$

trees T_n *with* n *labelled nodes, the* i^{th} *node having degree* $d(i)$.

This has been proved by Moon [5], [6] and Riordan [8].

LEMMA 2. *There are* $\binom{n-2}{k-1}(n-1)^{n-k-1}$ *trees* T_n *with* n *labelled nodes in which* $d(x) = k$ *for each node* x.

This was first proved by Clarke [1]; it follows easily from Lemma 1.

LEMMA 3. *If* $k = \left[\dfrac{(1+\varepsilon)\log n}{\log\log n}\right]$, *then* $\dfrac{n}{k!} < n^{-\varepsilon+o(1)}$ *as* $n \to \infty$, *for any positive constant* ε.

LEMMA 4. *If* $k = \left[\dfrac{(1-\varepsilon)\log n}{\log\log n}\right]$, *then* $\dfrac{n}{k!} > n^{\varepsilon+o(1)}$ *as* $n \to \infty$, *for any positive constant* ε.

LEMMA 5. *If* $k = [\log n]$, *then* $\dfrac{n}{k!} < n^2 \log n / n^{\log\log n}$ *for all sufficiently large values of* n.

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