MONOTONIC SINGULAR FUNCTIONS OF HIGH SMOOTHNESS

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This paper is concerned with the construction of monotonic functions that are *singular* (that is, have derivative zero almost everywhere) and possess good continuity properties as measured by the modulus of smoothness. We consider the operator Δ_h: $f(x) \to f(x+h) - f(x)$, and we recall that the *modulus of continuity* of f is the function

$$\omega(t) = \sup_{x} \sup_{0 \leq h \leq t} |\Delta_h f(x)|;$$

more generally, the (r^{th}-order) modulus of smoothness $\omega_r(t)$ is defined when Δ_h is replaced by the r^{th}-difference operator Δ_h^r. We shall refer to ω_2 simply as the *modulus of smoothness*.

Clearly, $\omega(t) = O(t)$ implies f is absolutely continuous. It is known that any bound on ω that does not imply $f \in \mathrm{Lip}\ 1$ is compatible with the existence of an increasing singular function whose modulus of continuity does not exceed ω. This seems implicit in a construction of F. Hausdorff (see [4, p. 30], also our paper [8]). *Added in proof.* The result was proved by P. Hartman and R. Kershner, *The structure of monotone functions*, Amer. J. Math. 59 (1937), 809-822 (see p. 818). We are indebted to P. L. Duren for this reference.

On the other hand, it is remarkable that the (Zygmund) class Z of functions for which $\omega_2(t) = O(t)$ contains increasing singular functions. This was first deduced in [2] (it underlies a long-known counterexample in the theory of conformal mapping). The first direct construction was given by G. Piranian [6]. Another construction, due to J.-P. Kahane, is mentioned without detailed verification in [6]. Piranian also outlines a proof that there is a singular function with $\omega_2(t) = o(t)$. Our main result (see Theorem 2) is the construction of an increasing singular function with $\omega_2(t) = O(t\,|\log t|^{-1/2})$, and this is essentially an unimprovable result (see the following paragraph). Our method is an adaptation of the basic idea (selective successive modifications) underlying Piranian's construction. The main novelty in our construction is the choice, as a "basic building block," of a trigonometric polynomial that vanishes very smoothly at the end points. Our choice has the two-fold advantage over Piranian's cubic polynomial that all of our successive approximations are twice differentiable (this simplifies the estimation of $\omega_2(t)$), and that for the proof of singularity we are able to invoke a known theorem on the convergence of lacunary trigonometric series. This construction, together with the extension to higher-order moduli of smoothness, is given in Section 1.

From the other side, M. Weiss and A. Zygmund [10] have shown that if $\omega_2(t) = O(t\,|\log t|^{-c})$ for some $c > 1/2$, then f is absolutely continuous and in fact has a derivative of class L^p for every $p < \infty$. Their proof is based on the theory of trigonometric series. They showed that their theorem becomes false for $c = 1/2$, by exhibiting a function f for which f' exists almost nowhere and

$$\omega_2(t) = O(t\,|\log t|^{-1/2}).$$

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