SPECIFIED RELATIONS IN THE IDEAL GROUP

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Introduction. Let $J = \sum Z x_i$ be the free abelian group based on the set $\{x_i\}$. We shall say that a subset I of elements of J satisfies condition α provided (i) all coefficients occurring on elements of I are nonnegative, (ii) to each finite subset $x_1, \cdots, x_k$ of $\{x_i\}$ and each finite set of nonnegative integers $n_1, \cdots, n_k$ there corresponds an element of I whose coefficient on x_i is n_i.

If A is a Dedekind domain and J is the divisor group of A (the free abelian group based on the primes of A), then the set I of integral principal divisors satisfies (i) by definition, and the weak-approximation theorem says that I satisfies (ii). Thus I satisfies condition α.

The main result of this paper provides a converse of the weak-approximation theorem (at least for the case where J has a countably infinite base). We shall prove a slight refinement (see Theorem 2.1) of the following assertion: If $J = \sum Z x_i$ is the free abelian group based on a countably infinite set $\{x_i\}$ and I is a subset of J that satisfies condition α, then there exists a Dedekind domain A such that the primes of A are in correspondence with the x_i in such a way that the principal divisors of A correspond to the elements of the subgroup generated by I.

This result fails for free groups of larger cardinality (we need a stronger hypothesis on I than condition α, and the proofs require transfinite techniques in almost every phase).

Section 1 of the present paper is devoted to some lemmas that are basically refinements of a technique, due to Goldman [3], for producing discrete valuations of specified types. In Section 2 we use these lemmas to give a proof of the theorem indicated above.

In Section 3 we give applications of the main theorem; we produce examples of 1) a Dedekind domain whose class group is cyclic of order n and all of whose prime ideals fall into one class, 2) a Dedekind domain A whose class group is isomorphic to Z, and with the property that each proper overring of A is a principal ideal domain, and 3) a Dedekind domain that is not an overring of the integral closure of a principal ideal domain (see [1, Example 1-9 and Remark 1-10 on page 61]).

Finally, in Section 4, we show that we can realize any finitely or countably infinitely generated class group by a Dedekind domain with finite residue class fields and unit group ± 1 (that is, in Goldman's sense, by a "special" Dedekind domain).

1. Throughout this section, A denotes a principal ideal domain subject to the four conditions

(1) A is countable,

(2) A has an infinite number of prime ideals,

(3) A/P is finite for all prime ideals $P \neq (0)$,

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