STRUCTURE THEOREMS FOR REGULAR LOCAL NOETHER LATTICES

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1. INTRODUCTION

The concept of a Noether lattice was introduced by R. P. Dilworth [2] as an abstration of the concept of the lattice of ideals of a Noetherian ring. A Noether lattice is a modular multiplicative lattice satisfying the ascending chain condition in which every element is a join of elements called principal elements. The principal elements are characterized by a pair of identities that are satisfied by the principal ideals of a ring. A generalization of Krull's principal-ideal theorem for Noether lattices states that the rank of a minimal prime containing a principal element is at most 1.

A Noether lattice is *local* if it has a unique proper maximal element. The definitions of dimension and rank carry over directly from Noetherian rings to Noether lattices. A local Noether lattice of dimension n is *regular* if its maximal element is a join of n principal elements. The structure of arbitrary regular local Noether lattices is closely related to a special class $\{RL_n\}$ of Noether lattices.

The elements of RL_n are those ideals of $F[x_1, \cdots, x_n]$ which are joins of products of the ideals $(x_1), (x_2), \cdots, (x_n)$. We show that RL_n is a sublattice of the lattice of ideals of $F[x_1, \cdots, x_n]$, and that it is a regular local Noether lattice. Our main results describe the relationship between $\{RL_n\}$ and arbitrary regular local Noether lattices as follows.

A local Noether lattice L *of dimension* n *is regular if and only if there exists a sublattice* L' *of* L *with the property that prime, primary, and principal elements in* L' *are, respectively, prime, primary, and principal in* L, *and* L' *is isomorphic to* RL_n.

A distributive regular local Noether lattice is isomorphic to one of the lattices RL_n.

In addition, we show that for $n \geq 2$, RL_n is not isomorphic to the lattice of ideals of any ring. In fact, an appropriate quotient sublattice of RL_2 provides an example of a Noether lattice for which the usual "converse" to Krull's principal-ideal theorem (a prime of rank 1 is a minimal prime of some principal ideal) does not hold.

2. PRELIMINARY DEFINITIONS AND RESULTS

The notation and terminology of this paper are the same as those of [2], with the exception that we use $\vee$ and $\wedge$ to denote the lattice operations, and $\leq$ to denote the lattice partical ordering, with $<$ reserved for proper inequality.

By a *multiplicative lattice* we mean a complete lattice L containing a unit element I and a null element 0, and provided with a commutative, associative, join-

Received September 11, 1967.