ON THE SYMPLECTIC BORDISM GROUPS OF THE SPACES Sp(n), HP(n), AND BSp(n)

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1. INTRODUCTION

Although the symplectic bordism ring $\Omega^{Sp} = \sum_{k=0}^{\infty} \Omega_k^{Sp}$ is largely undetermined, one may nonetheless investigate the symplectic bordism groups $\Omega_k^{Sp}(X)$ of spaces of importance in the study of symplectic bundles, manifolds, and so forth. In this paper, we carry out such a program for the symplectic groups Sp(n), their classifying spaces BSp(n), and the quaternionic projective spaces HP(n). We hope that the methods and results will lead to some measure of understanding of the symplectic bordism ring.

By a *symplectic manifold* we mean a compact smooth manifold together with a reduction of the structure group of its stable tangent bundle to the symplectic group. The *symplectic bordism group* $\Omega_k^{Sp}(X)$ of a space X consists of the bordism classes of pairs (M^k, f), where M^k is a closed symplectic k-manifold and $f: M^k \to X$ is a map; the bordism class of (M^k, f) is denoted by $[M^k, f]$ (see [2], [3]). $\Omega^{Sp}(X) = \sum_{k=0}^{\infty} \Omega_k^{Sp}(X)$ is a graded left module over the symplectic bordism ring $\Omega^{Sp} = \Omega^{Sp}(\text{point})$; put $[V^j][M^k, f] = [V^j \times M^k, F]$ with $F(v, m) = f(m)$.

There is a "fundamental class" homomorphism

$$\mu: \Omega_k^{Sp}(X) \to H_k(X)$$

from symplectic bordism to integral homology; $\mu[M^k, f] = f_*(\sigma_M)$, where $\sigma_M \in H_k(M^k)$ is the orientation class of the manifold. As in [2, p. 49] one may establish the following lemma.

1.1. LEMMA. *Let* X *be a* CW-*complex such that* $H_*(X)$ *is free abelian and* $\mu: \Omega^{Sp}(X) \to H_*(X)$ *is an epimorphism. Then* $\Omega^{Sp}(X)$ *is a free module over* Ω^{Sp}. *Moreover, if* $\{c_i\}$ *is a homogeneous basis for* $H_*(X)$ *and* $\{\gamma_i\}$ *are homogeneous elements of* $\Omega^{Sp}(X)$ *with* $\mu(\gamma_i) = c_i$, *then* $\{\gamma_i\}$ *is a basis for* $\Omega^{Sp}(X)$ *as a free* Ω^{Sp}-*module.*

If X is an H-space with multiplication $m: X \times X \to X$, then $\Omega^{Sp}(X)$ has a Pontrjagin ring structure. Namely, if we put

$$[M, f][M', f'] = [M \times M', m \circ (f \times f')],$$

then $\mu: \Omega^{Sp}(X) \to H_*(X)$ is a homomorphism of Pontrjagin rings. In case X is homotopy-commutative, $\Omega^{Sp}(X)$ has an anticommutative product

$$b \cdot a = (-1)^{\dim a \cdot \dim b} a \cdot b.$$

For the spaces mentioned in the title, we shall show that μ is onto, so that the symplectic bordism modules are free over Ω^{Sp}. Since Sp(n) and BSp(∞) are

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