ON THE ORDER OF A SIMPLY CONNECTED DOMAIN

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In an earlier paper [5] dealing with a problem about bounded analytic functions, I was led to associate with each bounded simply connected domain in the plane a certain countable ordinal number, called the *order* of the domain. The question was left open whether there actually exist domains of all possible orders. The purpose of the present note is to answer this question in the affirmative. The appropriate construction turns out to be almost embarrassingly easy. It nevertheless seems worth presenting, because it provides easily visualized examples of an interesting phenomenon connected with weak-star topologies.

In Section 1, we recall the basic definitions and describe the construction. The application to weak-star topologies is given in Section 2.

1. Let G be a bounded, simply connected domain in the plane. The *Carathéodory hull* (or *$\mathscr{C}$-hull*) of G is by definition the interior of the polynomially convex hull of G, that is, the interior of the set of points z_0 in the plane such that for all polynomials p,

$$|p(z_0)| \leq \sup\{|p(z)|: z \in G\}.$$

The $\mathscr{C}$-hull of G can be described in purely topological terms as the complement of the closure of the unbounded component of the complement of the closure of G. The components of a $\mathscr{C}$-hull are always simply connected.

If E is a bounded, simply connected domain containing G, then the *relative hull of* G *in* E, or the E-*hull* of G, is by definition the interior of the set of points z_0 in E such that for all functions f bounded and analytic in E,

$$|f(z_0)| \leq \sup\{|f(z)|: z \in G\}.$$

Relative hulls can also be described in purely topological terms: the E-hull of G is the interior of the set of points in E that cannot be separated from G by a crosscut of E (see [5]). The components of a relative hull are always simply connected.

For each countable ordinal number α we now define a simply connected domain G^α containing G. First we let G^1 be the component of the $\mathscr{C}$-hull of G that contains G. We then proceed by induction, assuming G^β has been defined for all $\beta < \alpha$. If α is not a limit ordinal, we let G^α be the component of the $G^{\alpha-1}$-hull of G that contains G. If α is a limit ordinal, we let G^α be the component of the interior of $\bigcap_{\beta<\alpha} G^\beta$ that contains G. From the topological description of relative hulls given above, it follows that if the inclusion $G^{\alpha+1} \subset G^\alpha$ is proper, then $G^\alpha - G^{\alpha+1}$ has interior points. Hence the inclusion is proper for at most countably many ordinals α, so that there is a least ordinal γ for which $G^\gamma = G^{\gamma+1}$. The ordinal γ is called the *order* of G.

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