GEOMETRIC CHARACTERIZATION OF DIFFERENTIABLE MANIFOLDS IN EUCLIDEAN SPACE, II

Herman Gluck

1. INTRODUCTION

In the predecessor [1] to this paper, I described a geometric characterization of one- and two-dimensional differentiable manifolds of class C^1 in Euclidean space. The general case will be given in this paper, which can be read independently of [1].

The one-dimensional result is the following.

PROTOTYPE. *Let* M *be a one-dimensional topological manifold in* R^n. *Then* M *is a* C^1*-manifold in* R^n *if and only if the secant map*

$$\Sigma\colon M \times M - \Delta \to P^{n-1}$$

admits a continuous extension over all of $M \times M$.

Here Δ denotes the diagonal $\{(x, x)\colon x \in M\}$ of $M \times M$, and the secant map Σ assigns to each pair (x, y) of distinct points of M the line through the origin in R^n (and thus an element of projective space P^{n-1}) that is parallel to the secant line through x and y.

The direct generalization of the secant map to the case of a k-dimensional manifold M in R^n would be a map Σ that assigns to each $(k+1)$-tuple $(x_0, x_1, \cdots, x_k)$ of linearly independent points of M the k-plane through the origin in R^n (and thus an element of the Grassmann manifold $G_{n,k}$) parallel to the secant k-plane through $x_0, x_1, \cdots, x_k$. This is fine.

The direct generalization of the prototype theorem would then say that a k-dimensional topological manifold M in R^n is a C^1-manifold if and only if the map Σ admits a continuous extension over the diagonal $\Delta = \{(x, x, \cdots, x)\colon x \in M\}$ of $M \times M \times \cdots \times M = (M)^{k+1}$. But this is incorrect!

The difficulty is already apparent in the two-dimensional case. Look at three linearly independent points x_0, x_1, x_2 that approach a single point x on a two-sphere. If the approach is "conventional" then the secant plane through x_0, x_1 and x_2 approaches the tangent plane to the two-sphere at x. But if x_0, x_1, x_2 and x all lie on an equator, then the secant plane through x_0, x_1 and x_2 contains the equator, and is therefore orthogonal to the tangent plane at x. So there is no hope for continuously extending the generalized secant map over the diagonal, even when the manifold is known to be differentiable.

What goes wrong? As the three points x_0, x_1, x_2 converge to x on the two-sphere, the three edges of the triangle $x_0 x_1 x_2$ approach the tangent plane at x. But the mere fact that the three edges of a triangle make small angles with a given plane does *not* imply that the triangle itself makes a small angle with the plane.

Received July 25, 1966.

Research on this paper was partially supported by a grant GP 4433 from the National Science Foundation. The author holds an Alfred P. Sloan Foundation Research Fellowship.