SINGULARITIES OF DIRICHLET SERIES WITH COMPLEX EXPONENTS

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1. INTRODUCTION

In this paper we are concerned with the function f(s) defined by

$$(1)\qquad f(s) = \sum_{n=1}^{\infty} a_n e^{-(\mu_n + i\nu_n)s},$$

where μ_n and ν_n are real numbers and μ_n increases and tends to infinity.

In order to ensure that the series (1) possesses an abscissa of convergence, and that this coincides with the abscissa of absolute convergence, we shall assume that

$$(2)\qquad \nu_n = o(\mu_n)$$

and that

$$(3)\qquad \frac{\log n}{\mu_n} \to 0 \text{ as } n \to \infty.$$

It is easy to see that if condition (3) is not satisfied, then the series (1) may fail to possess an abscissa of convergence, even if $\{\nu_n\}$ is bounded; as an example, we consider the series

$$(4)\qquad \sum_{1}^{\infty} \exp\{-[\log n + (-1)^n 2\pi i]s\}.$$

For $s = 1/4$, this series becomes $i\sum(-1)^n n^{-1/4}$, which converges; but for $s = 1/2$, the series becomes $-\sum n^{-1/2}$, and this diverges.

Our purpose in this paper is to extend to the series (1), where μ_n and ν_n satisfy (2) and (3), the theorem of Vivanti [3], that if the arguments of the coefficients of a power series all lie within a fixed angle less than π, then the positive point on the circle of convergence is a singularity of the sum of the series.

THEOREM 1. *Suppose that the series in (1) satisfies condition (3), that the abscissa of convergence* σ_c *is finite, and that*

$$(5)\qquad |\arg a_n - \sigma_c \nu_n| \le \chi < \pi/2,$$

where $-\pi < \arg a_n \le \pi$. *If* $\sigma_c = 0$, *suppose also that*

$$(6)\qquad |\nu_n| \le M,$$

where M *is some positive constant. Then* f(s) *has a singularity at* $s = \sigma_c$.

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