NORMS OF POWERS OF ABSOLUTELY CONVERGENT FOURIER SERIES IN SEVERAL VARIABLES

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In this paper we establish an upper bound for $\|f^n\|$, where f is an absolutely convergent Fourier series

$$f(\theta) = \sum_{\alpha} a_\alpha e^{i(\alpha,\theta)}$$

in k variables, with $\|f\| = \sum_\alpha |a_\alpha|$; here we use the notation $\alpha = (\alpha_1, \cdots, \alpha_k)$ for a k-tuple of integers, and we write $\theta = (\theta_1, \cdots, \theta_k)$ and $(\alpha, \theta) = \sum \alpha_j \theta_j$. We also use

$$D_j = \frac{\partial}{\partial \theta_j}, \qquad D^\beta = \prod_1^k D_j^{\beta_j}.$$

We introduce the partial ordering

$$\beta \geq \beta' \text{ if and only if } \beta_j \geq \beta_j' \text{ for } j = 1, \cdots, k.$$

Let $0 = (0, \cdots, 0)$ and $I = (1, \cdots, 1)$.

THEOREM. *Let* f *be given by an absolutely convergent Fourier series, and let* $|f(\theta)| \leq 1$ *for all* θ. *Suppose* $D^\beta f$ $(0 \leq \beta \leq I)$ *exists in the sense of Sobolev and belongs to* L_2. *Then*

$$\|f^n\| \leq M n^{k/2} \qquad (n = 1, 2, \cdots).$$

Remarks. For k = 1, the theorem was proved by Kahane (see [4, page 103]) by means of an inequality of F. Carlson [1]. We shall prove a generalization of Carlson's inequality (Lemma 2).

Kahane [3] showed that for k = 1 the estimate is the best possible estimate. His example is easily modified to show that

$$\|f^n\| \geq C n^{k/2} \qquad (C > 0,\ n = 1, 2, \cdots)$$

if $f(\theta) = e^{i\phi(\theta)}$ and ϕ is real, $\phi \in C^2$, and if for some θ the matrix $[D_h D_j\, \phi(\theta)]$ does not have zero as an eigenvalue. It is sufficient to deal with the localized problem, and we may rotate the coordinates to diagonalize the second derivatives (see [2]).

The proof is based on two lemmas. The first concerns polynomials in a complex variable $z = (z_1, \cdots, z_k)$. We use the notation $dz = dz_1 \cdots dz_k$ and $d\theta = d\theta_1 \cdots d\theta_k$.

LEMMA 1. *Let* $b_\alpha \geq 0$ $(\alpha \geq 0)$. *Suppose* $g(z) = \sum b_\alpha z^\alpha$ *is a polynomial. Then*

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