QUASICONFORMAL MAPPINGS OF THE UNIT DISC WITH TWO INVARIANT POINTS

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INTRODUCTION

Let Δ be the unit disc, and let $w = f(z)$ be a Q-quasiconformal mapping of Δ onto itself such that $f(0) = 0$ and $f(z_0) = z_0$ for some z_0 $(0 < |z_0| < 1)$. If $Q = 1$, then obviously $w = f(z)$ is the identity mapping. It is natural to ask how far a Q-quasiconformal mapping $w = f(z)$ satisfying the above-mentioned conditions can depart from the identity.

In this paper, we obtain a parametric representation for quasiconformal mappings of Δ onto itself that leave the points 0 and z_0 unchanged. Our results (Theorems 1 and 2) are analogues of corresponding results due to Tao-shing Shah [5]. A simple derivation of a parametric representation for quasiconformal mappings has recently been given by F. W. Gehring and E. Reich [3]. However, the variable complex dilatation as given by formula (2.1) in [3] does not imply the invariance of z_0 for changing t.

Theorems 1 and 2 enable us to obtain an estimate of $|f(z) - z|$ (Theorem 3) in terms of z, z_0, and Q for the class under consideration. In the limiting case, the estimate yields an inequality due to Tao-shing Shah [5].

1. THE CLASS $S_Q^{z_0}$ AND ITS SUBCLASSES

Let $S_Q^{z_0}$ denote the class of all functions f that map Δ onto itself Q-quasiconformally with $f(0) = 0$ and $f(z_0) = z_0$. Further, let S_* denote the class of all measurable complex dilatations μ defined a.e. in Δ and bounded by a constant less than 1. Let $(S)_*$ denote the subclass of S_* consisting of functions belonging to the class C^1 and continuable on $\overline{\Delta}$ as C^1-functions. Let $\hat{S}_*$ be the subclass of $(S)_*$ consisting of functions that have in $\overline{\Delta}$ partial derivatives of the first order subject to a global Hölder condition with a certain exponent δ $(0 < \delta \leq 1)$. Finally, let $(S)_Q^{z_0}$ and $\hat{S}_Q^{z_0}$ denote the subclasses of $S_Q^{z_0}$ consisting of functions generated by complex dilatations that belong to the classes $(S)_*$ and $\hat{S}_*$, respectively.

LEMMA 1. *The subclasses* $\hat{S}_Q^{z_0}$ *and* $(\hat{S})_Q^{z_0}$ *are dense in the class* $S_Q^{z_0}$.

The proof is analogous to the proofs in [1] and [4].

2. AN INTEGRAL LEMMA

In what follows, we consider functions f and the corresponding complex dilatations μ depending on one real parameter t.

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