TRANSFORMS OF CERTAIN MEASURES

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Let G be a locally compact, nondiscrete abelian group, and Γ its Pontrjagin dual. The Fourier-Stieltjes transform $\hat{\mu}$ of a measure μ is defined by the formula

$$\hat{\mu}(\gamma) = \int \overline{\gamma(x)}\,\mu(dx) \qquad (\mu \in M(G),\ \gamma \in \Gamma).$$

We present here generalizations of two theorems of Wik [2] concerning compact sets $P \subseteq G$ with the property that $\|\hat{\mu}\| \equiv \|\hat{\mu}\|_\infty = \|\mu\|$ for all $\mu \in M(P)$ (measures supported in P).

THEOREM 1. *If* $\limsup |\hat{\mu}| < \|\mu\|$ *for some* $\mu \in M(P)$, *then* $\|\hat{\lambda}\| < \|\lambda\|$ *for some* $\lambda \in M(P)$.

Here $\limsup |\hat{\mu}| = \inf_C \sup_{\gamma \notin C} |\hat{\mu}(\gamma)|$, the infimum being taken over all compact subsets C of Γ.

THEOREM 2. *Let* Γ_1 *be a closed subgroup of* Γ, *and let* Γ/Γ_1 *be compact. If*

(1) $\|\hat{\mu}\| = \|\mu\|$ *for all measures* $\mu \in M(P)$ *and*

(2) $\sup_{\gamma_1 \in \Gamma_1} |\hat{\sigma}(\gamma_1)| = \|\sigma\|$ *for all discrete measures* σ *in* P,

then

(3) $\sup_{\gamma_1 \in \Gamma_1} |\hat{\mu}(\gamma_1)| = \|\mu\|$ *for all measures in* M(P).

A general reference for the duality theory is Hewitt and Ross [1]; specific references are given below as needed. The author thanks the referee for pointing out a certain simplification in the proof of Theorem 1.

LEMMA 1. *For any measure* ν *concentrated on a countable subset* D *of* G, $\limsup |\hat{\nu}| = \|\hat{\nu}\|$.

Proof. Suppose that $\limsup |\hat{\nu}| < \|\hat{\nu}\|$; then $|\hat{\nu}(\gamma_0)| = \|\hat{\nu}\|$ for some $\gamma_0 \in \Gamma$. There exist a compact set $C \subseteq \Gamma$ and a positive number δ such that $|\hat{\nu}| < \|\hat{\nu}\| - \delta$ in the complement of C. Since ν is an atomic measure, there exist a finite set $\{d_1, d_2, \cdots, d_n\} \subseteq D$ and a positive number ε such that whenever

$$\gamma \in \Gamma \quad \text{and} \quad |\gamma(d_i) - 1| < \varepsilon \quad (1 \leq i \leq n),$$

then $|\hat{\nu}(\gamma + \gamma_0) - \hat{\nu}(\gamma_0)| < \delta$, whence $\gamma + \gamma_0 \in C$.

If χ is a character of G, not assumed to be continuous, then χ is in the pointwise closure of the set

$$C_1 = \{\gamma \in \Gamma: |\gamma(d_i) - \chi(d_i)| < \varepsilon/2,\ 1 \leq i \leq n\}$$

Received February 16, 1966.