REPRESENTATION RINGS

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1. INTRODUCTION

Let Λ be a ring with unity element. By a Λ-*module* we shall always mean a finitely generated unitary left Λ-module. If C is some category of Λ-modules, we may associate with C an abelian additive group a(C), generated by the set of symbols $\{[M]: M \in C\}$, with relations $[M] = [M'] + [M'']$ whenever $M \cong M' \oplus M''$. From this definition it follows at once that $[M] = [N]$ in a(C) if and only if there exists a module $X \in C$ such that $M \oplus X \cong N \oplus X$.

In particular, suppose that Λ is the group ring RG of a finite group G over an integral domain R. Take C to be the category of R-torsion-free RG-modules, and define multiplication in a(C) by means of

$$[M][N] = [M \otimes_R N] \qquad (M, N \in C).$$

Then a(C) becomes a commutative ring, hereafter denoted by a(RG) and called the *representation ring* of RG. Such rings have been studied in [4] to [7], and in [10].

Now let Z be the ring of rational integers, and let G be a group of order n. Define

$$Z' = \{a/b: a, b \in Z, (b, n) = 1\}.$$

Then Z' is a semilocal ring, useful in the study of indecomposable ZG-modules. The purpose of the present note is to investigate the relationship between the representation rings a(ZG) and a(Z'G), and to settle a conjecture raised at the end of [6].

Two Z-free ZG-modules M, N are said to lie in the same *genus* (notation: $M \vee N$) if and only if $Z' \otimes M \cong Z' \otimes N$. (The original definition of genus, as well as its equivalence with the above definition, may be found in [3]. See also [1, Section 81].)

In this note it will be shown that, as additive groups,

$$a(ZG) \cong b(ZG) \oplus a(Z'G), \tag{1}$$

where b(ZG) is some finite additive group which is an ideal in the ring a(ZG). Explicitly,

$$b(ZG) = \{[F] - [P]: F = \text{free ZG-module}, P \vee F\}. \tag{2}$$

We easily deduce that

$$b(ZG) = \{[ZG] - [P]: P \vee ZG\}, \tag{3}$$

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