ALMOST COMBINATORIAL MANIFOLDS AND THE ANNULUS CONJECTURE

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Dedicated to Professor Raymond L. Wilder on his seventieth birthday.

In this paper we investigate a class of conjectures related to the conjectures $\beta(n, m, k)$ in [5]. We prove (Corollary 3.5) that each member of a fairly large class of these conjectures is equivalent to the annulus conjecture. This equivalence leads to an example of a set in R^4 that could conceivably be wild, in which case the 4-dimensional annulus conjecture would be false. As a by-product of Section 3, some positive results are obtained; for example, in S^8, the union of a tame 6-cell D_1 and a tame 4-cell D_2 meeting in a 2-cell D is tame, provided D is locally flat in both Bd D_1 and Bd D_2.

In the first two sections we establish some results needed for the main "equivalence" theorems (Theorems 3.2 and 3.3). However, the results in these first sections are proved in greater generality than is needed later, and they are of independent interest.

Definitions. An n-manifold N is said to be an *almost-combinatorial* n-*manifold* (abbreviated AC n-*manifold*) provided that both Bd N and Int N support locally finite combinatorial structures; in this case, N will be assumed to be already equipped with such structures. The most notorious examples of almost-combinatorial manifolds are the fake annuli; a *fake* n-*annulus* is a manifold that is homeomorphic to the closure of a region in S^n bounded by a nonintersecting pair of locally flat (n - 1)-spheres in S^n. If one can triangulate a given fake annulus A, then it follows that A is homeomorphic to $S^{n-1} \times [0, 1]$. Thus, if one could triangulate every fake n-annulus, the annulus conjecture would be settled affirmatively in dimension n.

As usual, an *isotopy* of a space X is a collection h_t $(0 \leq t \leq 1)$ of homeomorphisms of X onto itself such that the map H: $X \times I \to X$ defined by $H(x, t) = h_t(x)$ is continuous. If f, g: $X \to Y$ are embeddings and $B \subset Y$, then f and g are *ambient isotopic leaving* B *fixed* if there exists an isotopy h_t $(0 \leq t \leq 1)$ of Y such that h_0 is the identity, $h_t \mid B$ is the identity for each t, and $h_1 f = g$.

Our definition of *locally tame* is the same as that used by Gluck in [9]. The concepts of *locally flat embedding* and *submanifold* are well known.

1. EMBEDDINGS INTO ALMOST-COMBINATORIAL MANIFOLDS

In the following, I denotes the unit interval [0, 1], and B stands for the set $\{0, 1\}$. If X is a space and F is an embedding of $X \times I$ into the manifold N such that $F(X \times B) \subset$ Bd N, we say that F *agrees with a collar structure for* Bd N *in a neighborhood of* $X \times B$ provided there exists a collaring G: Bd N $\times [0, 1/2) \to$ N (that is, an embedding such that $G(x, 0) = x$ for $x \in$ Bd N) and an $\varepsilon > 0$ such that

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