A COVERING-SPACE APPROACH TO RESIDUAL PROPERTIES OF GROUPS

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To R. L. Wilder, on his seventieth birthday.

INTRODUCTION

A group G is said to have a property P residually, or to be "residually P" if and only if for each $x \in G$ there is a normal subgroup V in G such that $x \notin V$ and G/V has the property P. This holds if and only if the intersection J(P) of the family $\mathscr{M}(P)$ of all such subgroups V is trivial. Algebraists have developed certain methods for proving that $J(P) = \{1\}$ for certain properties P; a good example is the theorem of Baumslag [5, p. 414], where P is the property of being finite. However, when G arises as the fundamental group of a path-connected topological space X (say a surface), it is of interest to look at the family of associated covering spaces X_A $(A \in \mathscr{M}(P))$, which form an inverse system with inverse limit X_*. Comparing this with X_J (where $J = J(P)$), we obtain our main result (Theorem 1 below), on the assumption that $\mathscr{M}$ has a linearly ordered cofinal subset relative to its ordering by inclusion; this theorem states that *there is a natural monomorphism* $H_1(X_J) \to H_1(X_*)$ *of Čech homology groups with compact carriers and arbitrary coefficients.*

Since X_* is an inverse limit, it is often fairly easy to compute $H_1(X_*)$; and if X (and hence X_J) is (say) locally contractible, then $H_1(X_J)$ is isomorphic to the singular homology group $H_1 S(X_J)$. The latter group, with coefficients in the group $\underset{\sim}{Z}$ of integers, is the abelianizer of J; therefore

the abelianizer of J *is isomorphic to a subgroup of* $H_1(X_*, \underset{\sim}{Z})$.

Thus, for example, if G is known not to possess perfect subgroups, then J is trivial if $H_1(X_*, \underset{\sim}{Z})$ is zero. Some applications are given in the last section (6) of this paper; for example, we show that the fundamental group of a surface is residually finite. The only previously known proof of this depends on the theory of Fuchsian groups (as a deduction from the theorem of Baumslag, quoted above). In Section 5, we prove our main theorem, using the somewhat technical Lemma 3 from Section 3. The remaining sections establish notation and gather standard requisite ideas about covering spaces, inverse limits, and Čech homology.

1. COVERING SPACES

We shall be interested in groups of the form $G = \pi_1(X, o)$, where X denotes a path-connected, LC^1, locally compact metric space with base-point o. Let $\mathscr{L}$ denote the lattice of all subgroups of G. Then with each A, $B \in \mathscr{L}$ such that $A \subseteq B$ there are associated based covering spaces and projection maps

(1.1) $$p_{AB}\colon (X_A, o_A) \to (X_B, o_B);$$

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