TWO THEOREMS ABOUT PERIODIC TRANSFORMATIONS OF THE 3-SPHERE

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To R. L. Wilder on his seventieth birthday.

Let T denote a transformation of period $\nu > 1$ of the 3-sphere Σ, and let Λ denote the set of fixed points of T. It is generally believed that *if* Λ *is a tame simple closed curve it must be unknotted,* but this conjecture of P. A. Smith has not yet been proved. In his thesis [4], C.H. Giffen proved that Λ can not be a torus knot. In this note I shall give a new and elementary proof of Giffen's theorem (Giffen's proof makes heavy use of fiber space theory), and then establish a condition that must be satisfied by the group Γ of Λ.

Suppose that Λ is a tame simple closed curve, and denote the orbit space Σ/T by S. Denote the collapsing map $\Sigma \to S$ by e and the image of Λ under e by L. It is known [10] that S is a closed 3-manifold, that L is a tame simple closed curve in S, and that e: $(\Sigma, \Lambda) \to (S, L)$ is a ν-fold cyclic covering, branched over L.

Since $e \mid \Sigma - \Lambda$: $\Sigma - \Lambda \to S - L$ is an unbranched ν-fold cyclic covering, the group $\Gamma = \pi(\Sigma - \Lambda)$ must be a normal subgroup of index ν of the group $G = \pi(S - L)$. It is known [6] that *the* 3-*manifold* S *must be simply connected.* Since this last statement has appeared in several places [2], [3], [4], [7], [8] without reference, I digress to give a brief proof of it.

Let m be a meridian of L—that is, an element of G represented by a meridian curve on the boundary of a tubular neighborhood of L. Since Λ is the branch curve of the ν-fold cyclic covering e, the element $\mu = m^\nu$ of Γ is a meridian of Λ. Since filling in the knot Λ maps Γ onto $\pi(\Sigma) = 1$, the consequence $\langle \mu \rangle$ of the element μ must be all of Γ, and hence $\langle m \rangle$, the consequence in G of m, must contain Γ. Since the elements $1, m, \cdots, m^{\nu-1}$ represent the ν cosets of Γ in G, it follows that $\langle m \rangle$ must be all of G. Now, filling in the knot L maps G onto $\pi(S)$. Since m is thereby mapped into 1, we see that $\pi(S) \approx G/\langle m \rangle = 1$. ∎

Let $\mathscr{A}(\Gamma)$ and $\mathscr{I}(\Gamma)$ denote the group of automorphisms and the group of inner automorphisms, respectively, and denote by $\mathscr{B}(\Gamma)$ the group of those automorphisms of Γ that induce the identity automorphism of Γ/Γ'. Thus an automorphism B of Γ belongs to $\mathscr{B}(\Gamma)$ if and only if each element γ of Γ has the same linking number with Λ as does its image $B(\gamma)$. Of course, $\mathscr{I}(\Gamma) \subset \mathscr{B}(\Gamma) \subset \mathscr{A}(\Gamma)$. The inner automorphism D_m: $g \to mgm^{-1}$ of G maps the normal subgroup Γ onto itself, and so it induces an automorphism Δ_m of Γ (which may or may not be an inner automorphism). Since the inner automorphism D_m^ν of G induces the inner automorphism Δ_μ: $\gamma \to \mu\gamma\mu^{-1}$, we see that $\Delta_m^\nu = \Delta_\mu \in \mathscr{I}(\Gamma)$. Since a loop in $\Sigma - \Lambda$ has linking number 0 with Λ if and only if the loop in $S - L$ into which it is projected by e has linking number 0 with L, we see that $\Gamma' = G'$; since D_m induces the identity automorphism of G/G', it follows that Δ_m induces the identity automorphism of $\Gamma/\Gamma' = \Gamma/G'$. Thus Δ_m belongs to the group $\mathscr{B}(\Gamma)$ of homologically faithful automorphisms.

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